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# Twisted boundary states in $c = 1$ coset conformal field theories

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Abstract: We study the mutual consistency of twisted boundary conditions in the coset conformal field theory $G/H$. We calculate the overlap of the twisted boundary states of $G/H$ with the untwisted ones, and show that the twisted boundary states are consistently defined in the charge-conjugation modular invariant. The overlap of the twisted boundary states is expressed by the branching functions of a twisted affine Lie algebra. As a check of our argument, we study the diagonal coset theory $so(2n)_1 \oplus so(2n)_1/so(2n)_2$, which is equivalent to the orbifold $S^1/\mathbb{Z}_2$ at a particular radius. We construct the boundary states twisted by the automorphisms of the unextended Dynkin diagram of $so(2n)$, and show their mutual consistency by identifying their counterpart in the orbifold. For the triality of $so(8)$, the twisted states of the coset theory correspond to neither the Neumann nor the Dirichlet boundary states of the orbifold and yield conformal boundary states that preserve only the Virasoro algebra.

Keywords: D-branes, Conformal Field Models in String Theory, Conformal and W Symmetry, Boundary Quantum Field Theory.



 

## Contents

## 1. Introduction

The nature of boundary conditions in rational conformal field theories (RCFTs) has been now fairly well understood [1]–[10]. There are a set of conditions that every consistent set of the boundary states should satisfy, which is called the sewing relations [2, 3]. The Cardy condition [1] is one of the sewing relations and expresses the consistency of the annulus amplitudes without insertion of vertex operators. It is now recognized that solving the Cardy condition is equivalent to finding the non-negative integer valued matrix representation (NIM-rep) of the fusion algebra[9, 11] under the assumption of the completeness of boundary states [4]. The classification problem of the possible boundary conditions in a given RCFT is therefore related with those of NIM-reps of the fusion algebra.

Since the fusion coefficients themselves form a representation of the fusion algebra, we always have at least one NIM-rep for each fusion algebra, which is realized by the fusion matrices and called the regular one. Therefore the point is whether we have a non-trivial NIM-rep other than the regular one. If we do not require an integer-valued representation, we have several one-dimensional ones realized by the (generalized) quantum dimensions, which in general take values in irrational numbers. However the requirement for a representation to be NIM is so severe that one finds a non-trivial NIM-rep only in limited cases. In these cases, the number of possible NIM-reps is greater than one and we have several solutions for one particular fusion algebra. Although these different NIM-reps are in general considered to be corresponding to different modular invariants, it is possible that two or more NIM-reps coexist in the same modular invariant and are compatible with each other.

This issue of mutual consistency of different NIM-reps has been studied in [8, 12] (see also [18]) for the case of the WZW models. In the WZW model based on a group $G$, the chiral algebra is an (untwisted) affine Lie algebra $\mathfrak{g}$, and the regular NIM-rep describes the boundary conditions preserving $\mathfrak{g}$. If the horizontal subalgebra of $\mathfrak{g}$ has an automorphism $\omega$, we can twist $\mathfrak{g}$ by $\omega$ to obtain a non-trivial boundary condition in the WZW model, and one can find a non-trivial NIM-rep corresponding to the twisted boundary condition. Eventually we obtain two types of NIM-reps in one theory: the regular and the twisted ones. The mutual consistency of these two has been shown by calculating the annulus amplitude between two types of the boundary states. The twisted affine Lie algebra associated with $\mathfrak{g}$ and $\omega$ plays an essential role in the calculation.

In this paper, we extend the analysis of the mutual consistency of different NIM-reps for the WZW models to an important class of CFTs, coset conformal field theories [13]. The boundary states in coset theories have been studied by several groups [14]–[23]. In particular, it has been shown that one can construct a NIM-rep of the $G/H$ theory starting from those of $G$ and $H$ [18, 23]. Hence a non-trivial NIM-rep of $G/H$ follows from the twisted NIM-reps of $G$ and $H$, if $G$ (or $H$) is equipped with an automorphism [14, 18]. We calculate the overlap of the corresponding twisted Cardy states of $G/H$ with the untwisted ones, and show that the twisted states are consistently defined in the charge-conjugation modular invariant. The overlap of the twisted boundary states is given by the branching functions of the twisted affine Lie algebras associated with the automorphism used for twisting the boundary condition. As a check of our argument, we study in detail the diagonal coset theory $so(2n)_1 \oplus so(2n)_1/so(2n)_2$, which is equivalent to the orbifold $S^1/\mathbb{Z}_2$ at a particular radius. We construct the boundary states twisted by the automorphisms of the unextended Dynkin diagram of $so(2n)$, and show their mutual consistency by identifying their counterpart in the orbifold. For the order-2 automorphism of $so(2n)$, the resulting states as well as the regular ones are realized by the usual Neumann and Dirichlet boundary states of the orbifold. For the triality of $so(8)$, however, the twisted states of the coset theory correspond to neither the Neumann nor the Dirichlet states. Rather they yield conformal boundary states that preserve only the Virasoro algebra. The conformal boundary states have been constructed for the case of $S^1$ [24]–[27]. Our twisted states provide their generalization to $S^1/\mathbb{Z}_2$.

The organization of this paper is as follows. We review some basic facts about the construction of NIM-reps in coset CFTs in the next section 2. In section 3, we argue the mutual consistency of different NIM-reps in coset theories. In particular, we show that the twisted Cardy states of $G/H$ are labeled by the branching functions of the associated twisted affine Lie algebras and have well-defined overlaps with the regular states. In section 4, we apply our method to the diagonal coset theory $so(2n)_1 \oplus so(2n)_1/so(2n)_2$ to obtain the Cardy states twisted by the automorphisms of $so(2n)$. We treat three types of automorphisms: the order-2 automorphism $\omega_2$ of $so(2n)$, the order-3 automorphism $\omega_3$ of $so(8)$ and the permutation automorphism $\pi$ of $so(2n) \oplus so(2n)$. For $\omega_2$ and $\omega_3$, we show that there exist non-trivial NIM-reps and give them in the explicit form. For the permutation automorphism $\pi$, however, the resulting NIM-rep is shown to yield nothing new. In section 5, using the equivalence between the $so(2n)$ coset theory and the orbifold $S^1/\mathbb{Z}_2$, we map the twisted states in the coset theory to the boundary states of the orbifold, in which the mutual consistency of the boundary states is well understood. We give some technical details in the appendices. The derivation of the conformal boundary states of $S^1/\mathbb{Z}_2$ is reported in appendix D.

## 2. Boundary states in coset theories

In this section, we review the construction of the Cardy states in coset theories developed in [18, 23].

### 2.1 Conventions

Let $\mathcal{A}$ be the chiral algebra of a RCFT, which is the Virasoro algebra or an extension thereof. We denote by $\mathcal{I}$ the set of all the possible irreducible representations of $\mathcal{A}$, and by $0 \in \mathcal{I}$ the vacuum representation. For the WZW model based on an affine Lie algebra $\mathfrak{g}$ at level $k$, $\mathcal{I}$ is given by the set $P^k_+(\mathfrak{g})$ of all the integrable highest-weight representations



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of $\mathfrak{g}$ at level $k$. Let $\mathcal{H}_\mu$ be the representation space for $\mu \in \mathcal{I}$. Then the character of the representation $\mu$ is written as

$$\chi_\mu(\tau) = \mathrm{Tr}_{\mathcal{H}_\mu}\, q^{L_0 - c/24}\,, \tag{2.1}$$

where $q = e^{2\pi i\tau}$ and $c$ is the central charge of the theory. The characters transform among themselves under the modular transformation $\tau \to -1/\tau$,

$$\chi_\mu(-1/\tau) = \sum_{\nu\in\mathcal{I}} S_{\mu\nu}\chi_\nu(\tau)\,. \tag{2.2}$$

The modular transformation matrix $S$ is a symmetric unitary matrix.

The elements of $\mathcal{I}$ form the fusion algebra $(\mu) \times (\nu) = \sum_{\rho\in\mathcal{I}} \mathcal{N}_{\mu\nu}{}^{\rho}(\rho)$. Here $\mathcal{N}_{\mu\nu}{}^{\rho}$ are non-negative integers called the fusion coefficients. It is often useful to introduce the matrices

$$(N_\mu)_\nu{}^\rho = \mathcal{N}_{\mu\nu}{}^{\rho}\,. \tag{2.3}$$

One can show that $N_\mu$ satisfy the fusion algebra

$$N_\mu N_\nu = \sum_{\rho\in\mathcal{I}} \mathcal{N}_{\mu\nu}{}^{\rho} N_\rho\,. \tag{2.4}$$

The Verlinde formula [28] relates the fusion coefficients $\mathcal{N}_{\mu\nu}{}^{\rho}$ with the modular transformation matrix $S$,

$$\mathcal{N}_{\mu\nu}{}^{\rho} = \sum_{\lambda\in\mathcal{I}} \frac{S_{\mu\lambda}S_{\nu\lambda}\overline{S_{\rho\lambda}}}{S_{0\lambda}} = \sum_{\lambda\in\mathcal{I}} S_{\nu\lambda}\gamma_\lambda^{(\mu)}\overline{S_{\rho\lambda}}\,, \tag{2.5}$$

where $\gamma_\lambda^{(\mu)}$ is the generalized quantum dimension

$$\gamma_\lambda^{(\mu)} = \frac{S_{\mu\lambda}}{S_{0\lambda}}\,. \tag{2.6}$$

In terms of the matrices $N_\mu$, the Verlinde formula can be written in the form

$$N_\mu = S\gamma^{(\mu)}S^\dagger\,, \qquad \gamma^{(\mu)} = \mathrm{diag}(\gamma_\nu^{(\mu)})_{\nu\in\mathcal{I}}\,. \tag{2.7}$$

The modular transformation matrix $S$ diagonalizes the fusion matrices $N_\mu$. The generalized quantum dimension $\{\gamma_\lambda^{(\mu)} \,|\, \lambda\,\text{fixed}\}$ is therefore a one-dimensional representation of the fusion algebra

$$\gamma_\lambda^{(\mu)}\gamma_\lambda^{(\nu)} = \sum_{\rho\in\mathcal{I}} \mathcal{N}_{\mu\nu}{}^{\rho}\gamma_\lambda^{(\rho)}\,. \tag{2.8}$$

A simple current $J$ of a RCFT is an element in $\mathcal{I}$ whose fusion with the elements in $\mathcal{I}$ induces a permutation $\mu \mapsto J\mu$ of $\mathcal{I}$ [29, 30],

$$(J) \times (\mu) = (J\mu)\,. \tag{2.9}$$

The set of all the simple currents forms an abelian group which we denote by $G_{\mathrm{sc}}$,

$$G_{\mathrm{sc}} = \{J \in \mathcal{I} \,|\, J:\ \text{simple current}\}. \tag{2.10}$$

$G_{\rm sc}$ is a multiplicative group in the fusion algebra. The group multiplication is defined by the fusion.

A simple current $J$ acts on the modular $S$ matrix as follows,

$$S_{J\lambda\,\mu} = S_{\lambda\mu} b_\mu(J)\,, \tag{2.11}$$

where $b_\mu(J)$ is the generalized quantum dimension of $J$,[1]

$$b_\mu(J) = \frac{S_{J\mu}}{S_{0\mu}} = \gamma_\mu^{(J)}\,. \tag{2.12}$$

In the matrix notation, the above relation can be written as

$$N_J S = S\, b(J)\,, \tag{2.13}$$

where $(N_J)_\mu{}^\nu = \mathcal{N}_{J\mu}{}^\nu = \delta_{J\mu\,\nu}$ and $b(J) = \mathrm{diag}(b_\mu(J))_{\mu\in\mathcal{I}}$. Since $\gamma_\mu^{(\lambda)}$ is a one-dimensional representation of the fusion algebra, $b_\mu(J)$ is a one-dimensional representation of $G_{\rm sc}$,

$$b_\mu(JJ') = b_\mu(J) b_\mu(J')\,, \qquad J, J' \in G_{\rm sc}\,. \tag{2.14}$$

Therefore $b_\mu(J)$ is of the form

$$b_\mu(J) = e^{2\pi i Q_\mu(J)}\,, \qquad Q_\mu(J) \in \mathbb{Q}\,, \tag{2.15}$$

since any one-dimensional representation of a finite group takes values in roots of unity. The phase $Q_\mu(J)$ is called the monodromy charge.

### 2.2 $G/H$ theories

The $G/H$ theory [13] is based on an embedding of the affine Lie algebra $\mathfrak{h}$ into $\mathfrak{g}$. A representation $\mu$ of $\mathfrak{g}$ is decomposed by representations $\nu$ of $\mathfrak{h}$ as follows,

$$\mathcal{H}_\mu = \bigoplus_\nu \mathcal{H}_{(\mu;\nu)} \otimes \mathcal{H}_\nu\,. \tag{2.16}$$

The primary fields of the $G/H$ theory are labeled by $(\mu;\nu)$ and the corresponding character is given by the branching functions. We denote by $\hat{\mathcal{I}}$ the set of all the possible primaries in the $G/H$ theory,

$$\hat{\mathcal{I}} = \{(\mu;\nu)|\, \mu \in \mathcal{I}^G, \nu \in \mathcal{I}^H, b_\mu^G(J) = b_\nu^H(J'), (J\mu; J'\nu) = (\mu;\nu), \forall (J,J') \in G_{\rm id}\}\,. \tag{2.17}$$

Here $\mathcal{I}^G$ ($\mathcal{I}^H$) is the set of all the primaries in the $G$ ($H$) theory. $G_{\rm id} \subset G_{\rm sc}^G \times G_{\rm sc}^H$ is the group of the identification currents corresponding to the common center of $G$ and $H$. $b_\mu^G(J)$ ($b_\nu^H(J')$) expresses the monodromy charge of the $G$ ($H$) theory. The condition $b_\mu^G(J) = b_\nu^H(J')$ is the selection rule of the branching of representations, while the relation $(J\mu; J'\nu) = (\mu;\nu)$ is the field identification.

[1] We write $J0$ as $J$ since the action of $J$ on 0 yields $J$ itself.

In this paper, we restrict ourselves to the case that all the identification orbit have the same length $N_0 = |G_{\rm id}|$,

$$N_0 = |\{(J\mu, J'\nu) \in \mathcal{I}^G \otimes \mathcal{I}^H |\, (J,J') \in G_{\rm id}\}| = |G_{\rm id}|\,. \tag{2.18}$$

In other words, there is no fixed point in the field identification [31]

$$(J\mu, J'\nu) \neq (\mu,\nu)\,, \qquad \forall(\mu,\nu) \in \mathcal{I}^G \otimes \mathcal{I}^H\,, \qquad \forall(J,J') \neq 1 \in G_{\rm id}\,. \tag{2.19}$$

The character of the coset theory is the branching function $\chi_{(\mu;\nu)}$ of the algebra embedding $h \subset g$. From the branching rule (2.16), we find

$$\chi^G_\mu = \sum_\nu \chi_{(\mu;\nu)} \chi^H_\nu\,. \tag{2.20}$$

The modular transformation $S$-matrix of the $G/H$ theory reads

$$\hat{S}_{(\mu;\nu)(\mu';\nu')} = N_0 S^G_{\mu\mu'} \overline{S^H_{\nu\nu'}} = N_0 S^G_{\mu\mu'} S^H_{\nu\bar{\nu'}}\,, \tag{2.21}$$

where $N_0$ is the length of the identification orbit (2.18) and $S^G$ ($S^H$) is the $S$-matrix of the $G$ ($H$) theory. $\bar{\nu'}$ is the representation conjugate to $\nu'$.

The fusion coefficients of the coset theory can be calculated via the Verlinde formula (2.5),

$$\begin{aligned}
\hat{\mathcal{N}}_{(\mu;\nu)(\mu';\nu')}{}^{(\mu'';\nu'')} &= \sum_{(\rho;\sigma)\in\hat{\mathcal{I}}} \frac{\hat{S}_{(\mu;\nu)(\rho;\sigma)} \hat{S}_{(\mu';\nu')(\rho;\sigma)} \overline{\hat{S}_{(\mu'';\nu'')(\rho;\sigma)}}}{\hat{S}_{(0;0)(\rho;\sigma)}} \\
&= \frac{1}{N_0} \sum_{(\rho,\sigma)\in\mathcal{I}^G\otimes\mathcal{I}^H} \frac{1}{N_0} \sum_{(J,J')\in G_{\rm id}} b^G_\rho(J) b^H_\sigma(J')^{-1} \times \\
&\qquad\qquad \times N_0^2 \frac{S^G_{\mu\rho} S^G_{\mu'\rho} \overline{S^G_{\mu''\rho}}}{S^G_{0\rho}} \frac{S^H_{\nu\bar{\sigma}} S^H_{\nu'\bar{\sigma}} \overline{S^H_{\nu''\bar{\sigma}}}}{S^H_{0\bar{\sigma}}} \\
&= \sum_{(J,J')\in G_{\rm id}} \mathcal{N}^G_{\mu\, J\mu'}{}^{\mu''} \mathcal{N}^H_{\nu\, J'\nu'}{}^{\nu''}\,.
\end{aligned} \tag{2.22}$$

Here we used our assumption of no fixed points to rewrite the sum

$$\sum_{(\rho;\sigma)\in\hat{\mathcal{I}}} \to \frac{1}{N_0} \sum_{(\rho,\sigma)\in\mathcal{I}^G\otimes\mathcal{I}^H} \frac{1}{N_0} \sum_{(J,J')\in G_{\rm id}} b^G_\rho(J) b^H_\sigma(J')^{-1}\,. \tag{2.23}$$

The projection operator introduced above takes account of the selection rule.

### 2.3 NIM-reps in RCFTs

In the bulk theory, we have a pair of algebras $\mathcal{A}$ and $\tilde{\mathcal{A}}$, which correspond to the holomorphic and the anti-holomorphic sectors, respectively. The space $\mathcal{H}$ of the states in the bulk theory can therefore be decomposed into the representations of $\mathcal{A} \otimes \tilde{\mathcal{A}}$ as

$$\mathcal{H} = \bigoplus_{\mu,\nu\in\mathcal{I}} M_{\mu\nu}\, \mathcal{H}_\mu \otimes \widetilde{\mathcal{H}}_{\bar{\nu}} \qquad (M_{\mu\nu} \in \mathbb{Z}_{\geq 0})\,. \tag{2.24}$$

The modular invariant partition function associated with this spectrum reads

$$Z(\tau) = \sum_{\mu,\nu\in\mathcal{I}} M_{\mu\nu}\,\chi_\mu(\tau)\overline{\chi_\nu(\tau)}\,. \tag{2.25}$$

In this paper, we restrict ourselves to the case of the charge-conjugation invariant $M_{\mu\nu} = \delta_{\mu\nu}$.

In the presence of boundaries, $\mathcal{A}$ and $\tilde{\mathcal{A}}$ are related with each other via appropriate boundary conditions. In the closed string channel, we can describe boundary conditions in terms of boundary states, which satisfy the following gluing condition

$$(\omega(W_n) - (-1)^h \widetilde{W}_{-n})|\alpha\rangle_\omega = 0\,. \tag{2.26}$$

Here $W$ and $\widetilde{W}$ are the currents of $\mathcal{A}$ and $\tilde{\mathcal{A}}$, respectively. $h$ is the conformal dimension of the currents and $\omega$ is an automorphism of the chiral algebra $\mathcal{A}$. $\omega$ induces an isomorphism $\mu \mapsto \omega\mu$ of the representations $\mu \in \mathcal{I}$ of the chiral algebra $\mathcal{A}$. In order to keep the conformal invariance, $\omega$ has to fix the generators $L_n$ of the Virasoro algebra,

$$\omega(L_n) = L_n\,. \tag{2.27}$$

In the following, a boundary condition for $\omega \neq 1$ will be called the *twisted* type, while $\omega = 1$ the *untwisted* type.

A generic boundary state $|\alpha\rangle$ satisfying the boundary condition (2.26) is a linear combination of the Ishibashi states [32]

$$|\alpha\rangle_\omega = \sum_{\mu\in\mathcal{E}} {\psi_\alpha}^\mu \frac{1}{\sqrt{S_{0\mu}}}|\mu\rangle\!\rangle_\omega \qquad (\alpha\in\mathcal{V})\,. \tag{2.28}$$

Here $\mathcal{E}$ is a set labeling the Ishibashi states compatible with the gluing automorphism $\omega$ and the modular invariant $Z$, while $\mathcal{V}$ is a set labeling the boundary states. For the charge-conjugation modular invariant, $\mathcal{E}$ is given by the set $\mathcal{I}^\omega$ of all the representations fixed by $\omega$,

$$\mathcal{I}^\omega = \{\mu\in\mathcal{I}\,|\,\omega\mu = \mu\}\,. \tag{2.29}$$

We normalize the Ishibashi states as follows,

$${}_\omega\langle\!\langle\mu|q^{H_c}|\nu\rangle\!\rangle_\omega = \delta_{\mu\nu}\chi_\mu(\tau) \quad (\mu,\nu\in\mathcal{I}^\omega)\,, \tag{2.30}$$

where $H_c = \frac{1}{2}(L_0 + \tilde{L}_0 - \frac{c}{12})$ is the closed string hamiltonian.

The coefficients ${\psi_\alpha}^\mu$ together with the sets $\mathcal{E}$ and $\mathcal{V}$ should be chosen appropriately for the mutual consistency of the boundary states [1, 9]. Consider the annulus amplitude between two boundary states $|\alpha\rangle$ and $|\beta\rangle$,

$$Z_{\alpha\beta} = \langle\beta|\tilde{q}^{H_c}|\alpha\rangle = \sum_{\nu\in\mathcal{E},\mu\in\mathcal{I}} {\psi_\alpha}^\nu \frac{S_{\mu\nu}}{S_{0\nu}}\overline{{\psi_\beta}^\nu}\,\chi_\mu(\tau) = \sum_{\mu\in\mathcal{I}} {n_{\mu\alpha}}^\beta\chi_\mu(\tau) \qquad (\alpha,\beta\in\mathcal{V})\,. \tag{2.31}$$

Here we denote by ${n_{\mu\alpha}}^\beta$ the multiplicity of the representation $\mu\in\mathcal{I}$ in $Z_{\alpha\beta}$,

$${n_{\mu\alpha}}^\beta = \sum_{\nu\in\mathcal{E}} {\psi_\alpha}^\nu \gamma_\nu^{(\mu)}\overline{{\psi_\beta}^\nu}\,. \tag{2.32}$$

In the matrix notation, this can be written as

$$n_\mu = \psi\gamma^{(\mu)}\psi^\dagger, \quad (n_\mu)_\alpha{}^\beta = n_{\mu\alpha}{}^\beta, \quad (\psi)_\alpha{}^\mu = \psi_\alpha{}^\mu\,. \tag{2.33}$$

Clearly, $n_\mu$ are matrices with non-negative integer entries for consistent boundary states. In particular, $n_0 = 1$ for the uniqueness of the vacuum in the open string channel. This condition for the consistency of the annulus amplitude is called the Cardy condition and the boundary states satisfying the Cardy condition are called the Cardy states. Under the assumption of the completeness $|\mathcal{V}| = |\mathcal{E}|$ [4], the Cardy condition implies that $\psi$ is a unitary matrix. The matrices $n_\mu$ are therefore related with the generalized quantum dimensions via a similarity transformation (2.33) and satisfy the fusion algebra

$$n_\mu n_\nu = \sum_{\rho\in\mathcal{I}} \mathcal{N}_{\mu\nu}{}^\rho n_\rho\,, \qquad n_\mu^T = n_{\bar\mu}\,. \tag{2.34}$$

Since $n_{\mu\alpha}{}^\beta$ take values in $\mathbb{Z}_{\geq 0}$, $\{n_\mu\}$ forms a non-negative integer matrix representation (NIM-rep) of the fusion algebra [9]. For each set of the mutually consistent boundary states, we have a NIM-rep $\{n_\mu |\, \mu\in\mathcal{I}\}$ of the fusion algebra. The simplest example is the regular NIM-rep

$$\psi = S \qquad (\mathcal{E} = \mathcal{V} = \mathcal{I})\,, \qquad n_\mu = N_\mu\,, \tag{2.35}$$

which corresponds to the trivial gluing automorphism $\omega = 1$ in the charge-conjugation modular invariant and exists in any RCFT. We note that there are many 'unphysical' NIM-reps that do not correspond to any modular invariant [11]. This fact shows that the Cardy condition is not a sufficient but a necessary condition for consistency.

The similarity between $S_{\mu\nu}$ and $\psi_\alpha{}^\mu$ (2.7) and (2.33) suggests that the simple currents act also on a diagonalization matrix $\psi$. As is discussed in [11, 18, 12, 23], we actually have two types of actions of the simple currents on $\psi$,

$$\psi_{J\alpha}{}^\mu = \psi_\alpha{}^\mu\, b_\mu(J)\,, \qquad n_J\psi = \psi\, b(J)\,, \qquad J\in G(\mathcal{V})\,, \tag{2.36a}$$

$$\psi_\alpha{}^{J\mu} = \tilde b_\alpha(J)\,\psi_\alpha{}^\mu\,, \qquad \psi N_J^T = \tilde b(J)\psi\,, \qquad J\in G(\mathcal{E})\,. \tag{2.36b}$$

Here $G(\mathcal{V})$ and $G(\mathcal{E})$ are the groups of simple currents for $\mathcal{V}$ and $\mathcal{E}$,

$$G(\mathcal{V}) = G_{\rm sc}/\mathcal{S}(\mathcal{V})\,, \tag{2.37a}$$

$$G(\mathcal{E}) = \{J\in G_{\rm sc}\,|\,J:\mathcal{E}\to\mathcal{E}\}\,, \tag{2.37b}$$

where $\mathcal{S}(\mathcal{V})$ is the stabilizer of $\mathcal{V}$, i.e. a subgroup of $G_{\rm sc}$ which acts trivially on $\mathcal{V}$,

$$\mathcal{S}(\mathcal{V}) = \{J_0\in G_{\rm sc}\,|\,J_0\alpha = \alpha,\,\forall\alpha\in\mathcal{V}\} = \{J_0\in G_{\rm sc}\,|\,b_\mu(J_0) = 1,\,\forall\mu\in\mathcal{E}\}\,. \tag{2.38}$$

$\tilde b_\alpha(J)$ are the counterpart of $b_\mu(J)$ and defined as

$$\tilde b_\alpha(J) = \frac{\psi_\alpha{}^J}{\psi_\alpha{}^0}\,. \tag{2.39}$$

The group $G(\mathcal{V})$ acts on $\mathcal{V}$ as a permutation of the Cardy states, and the group $G(\mathcal{E})$ enables us to define the "charge" $\tilde b_\alpha$ of the Cardy states, which takes values in the dual group of $G(\mathcal{E})$. Clearly, for the regular NIM-rep $\psi = S$, these two groups coincide with the simple current group $G_{\rm sc}$ itself.



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The transformation property (2.36a) follows from the fact that $n_J$ is a permutation matrix on $\mathcal{V}$ [11, 18]. The second one (2.36b) is related [11] with the following algebra [4, 5, 33, 9]

$$\frac{\psi_\alpha{}^\mu}{\psi_\alpha{}^0}\frac{\psi_\alpha{}^\nu}{\psi_\alpha{}^0} = \sum_{\rho\in\mathcal{E}} \mathcal{M}_{\mu\nu}{}^\rho \frac{\psi_\alpha{}^\rho}{\psi_\alpha{}^0}\,, \tag{2.40}$$

which coincides in some cases with the algebra that appeared in the classification of bulk theories [34]. Actually, if we set

$$(M_\mu)_\nu{}^\rho = \mathcal{M}_{\mu\nu}{}^\rho, \quad \tilde{\gamma}^{(\mu)} = \mathrm{diag}\left(\frac{\psi_\alpha{}^\mu}{\psi_\alpha{}^0}\right)_{\alpha\in\mathcal{V}}, \tag{2.41}$$

the algebra (2.40) can be written in the form

$$\psi M_\mu^T = \tilde{\gamma}^{(\mu)}\psi\,. \tag{2.42}$$

Hence, the transformation property (2.36b) follows from eq. (2.40) if $M_J(J \in G_{\mathrm{id}}(\mathcal{E}))$ coincides with $N_J$ restricted on $\mathcal{E}$.

We are now ready to state the construction of NIM-reps in $G/H$ theories [18, 23]. The starting point of the construction is a pair of NIM-reps $(\psi^G, \psi^H)$ of the $G$ and the $H$ theories with the sets $\mathcal{V}^G, \mathcal{E}^G, \mathcal{V}^H$ and $\mathcal{E}^H$.[2] Then one can make a NIM-rep of the $G/H$ theory in the following form

$$\hat{\psi}_{(\alpha;\beta)}{}^{(\mu;\nu)} = \sqrt{|G_{\mathrm{id}}(\mathcal{E})||G_{\mathrm{id}}(\mathcal{V})|}\,\psi_\alpha^{G\mu}\psi_\beta^{H\bar{\nu}}\,, \qquad (\alpha;\beta)\in\hat{\mathcal{V}},\ (\mu;\nu)\in\hat{\mathcal{E}}\,. \tag{2.43}$$

Here, $\hat{\mathcal{V}}$ and $\hat{\mathcal{E}}$ are labels of the Cardy states and the Ishibashi states of $G/H$, respectively, and defined as

$$\begin{aligned}
\hat{\mathcal{V}} &= \{(\alpha;\beta)\,|\,\alpha\in\mathcal{V}^G, \beta\in\mathcal{V}^H,\, \tilde{b}_\alpha^G(J) = \tilde{b}_\beta^H(J')\,(\forall(J,J')\in G_{\mathrm{id}}(\mathcal{E})),\\
&(J\alpha; J'\beta) = (\alpha;\beta)\,(\forall(J,J')\in G_{\mathrm{id}}(\mathcal{V}))\}, &(2.44)\\
\hat{\mathcal{E}} &= \{(\mu;\nu)\,|\,\mu\in\mathcal{E}^G, \nu\in\mathcal{E}^H,\, b_\mu^G(J) = b_\nu^H(J')\,(\forall(J,J')\in G_{\mathrm{id}}(\mathcal{V})),\\
&(J\mu; J'\nu) = (\mu;\nu)\,(\forall(J,J')\in G_{\mathrm{id}}(\mathcal{E}))\}. &(2.45)
\end{aligned}$$

$G_{\mathrm{id}}(\mathcal{E})$ and $G_{\mathrm{id}}(\mathcal{V})$ are the groups of the identification currents for the Ishibashi states and the Cardy states, respectively,

$$G_{\mathrm{id}}(\mathcal{E}) = G_{\mathrm{id}} \cap (G(\mathcal{E}^G)\times G(\mathcal{E}^H))\,, \tag{2.46}$$
$$G_{\mathrm{id}}(\mathcal{V}) = G_{\mathrm{id}}/(G_{\mathrm{id}}\cap\widehat{\mathcal{S}}(\mathcal{V}))\,, \tag{2.47}$$

where $\widehat{\mathcal{S}}(\mathcal{V})$ is the stabilizer for the action of the simple currents on the Cardy states[3]

$$\widehat{\mathcal{S}}(\mathcal{V}) = \{(J,J')\in G_{\mathrm{sc}}|\, b_\mu^G(J) = b_\nu^H(J')\ \forall(\mu,\nu)\in\mathcal{E}^G\otimes\mathcal{E}^H\}\,. \tag{2.48}$$

[2] This can be generalized to the case of NIM-reps not factorizable into the $G$ and the $H$ parts [23].
[3] In [23], $\widehat{\mathcal{S}}(\mathcal{V})$ is denoted by $\mathcal{S}^c(\mathcal{V}^c)$.

Let $n^G(n^H)$ be the NIM-rep matrix corresponding to $\psi^G(\psi^H)$. Then one can express the NIM-rep matrix $\hat{n}_{(\mu;\nu)}$ of the $G/H$ theory in terms of $n^G_\mu$ and $n^H_\nu$ as follows

$$(\hat{n}_{(\mu;\nu)})_{(\alpha;\beta)}{}^{(\alpha';\beta')} = \sum_{(J,J')\in G_{\rm id}(\mathcal{V})} (n^G_\mu)_{J\alpha}{}^{\alpha'} (n^H_\nu)_{J'\beta}{}^{\beta'} . \tag{2.49}$$

For the regular NIM-reps, $\psi^G = S^G$, $\psi^H = S^H$, one obtains $G_{\rm id}(\mathcal{E}) = G_{\rm id}(\mathcal{V}) = G_{\rm id}$ and $\hat{\psi}$ coincides with $\hat{S}$.

In [18, 23], the condition $\tilde{b}^G_\alpha(J) = \tilde{b}^H_\beta(J')$ is called the brane selection rule, while $(J\alpha; J'\beta) = (\alpha;\beta)$ is called the brane identification. If there is no fixed point in the brane identification, the matrix (2.43) yields a NIM-rep $\hat{n}_{(\mu;\nu)}$ of the $G/H$ theory. If we have fixed points in the brane identification, we have to resolve them in order to obtain a unitary $\hat{\psi}$. This is the same situation as the field identification fixed points [31].

## 3. Mutual consistency of twisted boundary conditions

As we have seen in the previous section, the problem of solving the Cardy conditions is equivalent to finding an appropriate NIM-rep of the fusion algebra. However, the number of solutions may be greater than one and we may have several NIM-reps for one particular fusion algebra. Although these different NIM-reps are in general considered to be corresponding to different modular invariants, it is possible that two or more NIM-reps coexist in the same modular invariant and are compatible with each other. This issue of mutual consistency of different NIM-reps has been studied in [8, 12] (see also [18]) for the case of the boundary states twisted by an automorphism of the chiral algebra. We briefly review this in the following and discuss its application to the case of coset models.

We begin with the relation of the twisted Ishibashi state $|\mu\rangle\rangle_\omega$ for $\omega \neq 1$ with the untwisted one $|\mu\rangle\rangle$. Introducing an orthonormal basis $\{|\varphi\rangle\}$ of $\mathcal{H}_\mu$, the untwisted Ishibashi state can be written as

$$|\mu\rangle\rangle = \sum_\varphi |\varphi\rangle \otimes \overline{|\varphi\rangle} \in \mathcal{H}_\mu \otimes \widetilde{\mathcal{H}}_{\bar{\mu}} . \tag{3.1}$$

One can construct the Ishibashi states for $\omega \neq 1$ starting from $|\mu\rangle\rangle$. Let $\Omega$ be a unitary operator mapping $\mathcal{H}_\mu$ to $\mathcal{H}_{\omega\mu}$,

$$\Omega : \mathcal{H}_\mu \to \mathcal{H}_{\omega\mu} , \qquad \Omega W_n \Omega^\dagger = \omega(W_n) , \qquad \Omega \widetilde{W}_n \Omega^\dagger = \widetilde{W}_n . \tag{3.2}$$

Then the twisted Ishibashi state $|\mu\rangle\rangle_\omega$ can be written as

$$|\mu\rangle\rangle_\omega = \Omega|\mu\rangle\rangle_{\omega=1} = \sum_\varphi (\Omega|\varphi\rangle) \otimes \overline{|\varphi\rangle} \in \mathcal{H}_{\omega\mu} \otimes \widetilde{\mathcal{H}}_{\overline{\mu}} . \tag{3.3}$$

Actually, one can check that this state satisfies the twisted gluing condition,

$$\begin{aligned} (\omega(W_n) - (-1)^h \widetilde{W}_{-n})|\mu\rangle\rangle_\omega &= (\Omega W_n \Omega^\dagger - (-1)^h \widetilde{W}_{-n})\Omega|\mu\rangle\rangle_{\omega=1} \\ &= \Omega(W_n - (-1)^h \widetilde{W}_{-n})|\mu\rangle\rangle_{\omega=1} \\ &= 0 . \end{aligned} \tag{3.4}$$

The normalization of $|\mu\rangle\rangle_\omega$ follows from that of $|\mu\rangle\rangle$ and coincides with our convention (2.30),

$$ {}_\omega\langle\langle\mu|q^{H_c}|\mu'\rangle\rangle_\omega = \langle\langle\mu|\Omega^\dagger q^{H_c}\Omega|\mu'\rangle\rangle = \langle\langle\mu|q^{H_c}|\mu'\rangle\rangle = \delta_{\mu\mu'}\chi_\mu(\tau)\,. \tag{3.5} $$

We will also need the overlap of the twisted Ishibashi state with the untwisted one. Since $|\mu\rangle\rangle_\omega$ couples $\omega\mu$ with $\bar{\mu}$, we have a non-vanishing overlap if and only if $\mu \in \mathcal{I}^\omega$,

$$ \langle\langle\mu|q^{H_c}|\mu'\rangle\rangle_\omega = \langle\langle\mu|q^{H_c}\Omega|\mu'\rangle\rangle = \begin{cases} \chi_\mu^\omega(\tau) & \text{if } \mu = \mu' \in \mathcal{I}^\omega\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{3.6} $$

Here $\chi_\mu^\omega$ is the twining character [35]

$$ \chi_\mu^\omega(\tau) = \mathrm{Tr}_{\mathcal{H}_\mu}\,\Omega q^{L_0 - c/12}\,. \tag{3.7} $$

For the case of (untwisted) affine Lie algebras, the twining character coincides with the ordinary character of a twisted affine Lie algebra called the orbit Lie algebra [35].

In order to obtain the spectrum in the open string channel, we need the modular transformation of the twining character. Since the twining character appears in the overlap between twisted and untwisted states, its modular transformation is written in terms of the characters of the twisted chiral algebra $\mathcal{A}^\omega$. Here we denote by $\mathcal{A}^\omega$ an algebra generated by the currents with the twisted boundary condition $W(e^{2\pi i}z) = \omega(W(z))$. Let $\tilde{\mathcal{I}}$ be the set of all the possible representations of the twisted chiral algebra $\mathcal{A}^\omega$ and $\chi_{\tilde{\mu}}$ ($\tilde{\mu} \in \tilde{\mathcal{I}}$) be the corresponding character. Then the modular transformation of the twining character can be written as

$$ \chi_\mu^\omega\left(\frac{-1}{\tau}\right) = \sum_{\tilde{\mu}\in\tilde{\mathcal{I}}} \tilde{S}^{(0)}_{\mu\tilde{\mu}}\,\chi_{\tilde{\mu}}\left(\frac{\tau}{r}\right) \qquad (\mu \in \mathcal{I}^\omega)\,, \tag{3.8a} $$

$$ \chi_{\tilde{\mu}}\left(\frac{-1}{\tau}\right) = \sum_{\mu\in\mathcal{I}^\omega} \tilde{S}^{(1)}_{\tilde{\mu}\mu}\,\chi_\mu^\omega\left(\frac{\tau}{r}\right) \qquad (\tilde{\mu} \in \tilde{\mathcal{I}})\,, \tag{3.8b} $$

where $\tilde{S}^{(0)}$ and $\tilde{S}^{(1)}$ are some unitary matrices [8]. We divide the argument of $\chi_{\tilde{\mu}}$ by the order $r$ of the automorphism $\omega$ so that $\chi_{\tilde{\mu}}(\tau)$ is expanded in integer powers of $q = e^{2\pi i\tau}$. The properties of $\tilde{S}^{(0)}$ and $\tilde{S}^{(1)}$ have been investigated in [8]. In particular, it has been shown that $\tilde{S}^{(1)}$ is related with $\tilde{S}^{(0)}$ via transposition,

$$ \tilde{S}^{(1)}_{\tilde{\mu}\mu} = \eta_\mu^{-2}\tilde{S}^{(0)}_{\mu\tilde{\mu}}\,, \tag{3.9} $$

where $\eta_\mu$ is an appropriate phase factor.

If $\mathcal{A}$ is an (untwisted) affine Lie algebra $\mathfrak{g}^{(1)}$ at level $k$, the corresponding twisted chiral algebra $\mathcal{A}^\omega$ is the twisted affine Lie algebra $\mathfrak{g}^{(r)}$ at level $k$ and $\tilde{\mathcal{I}} = P_+^k(\mathfrak{g}^{(r)})$. Since the modular transformation of the character of $\mathfrak{g}^{(r)}$ is expressed by the characters of another affine Lie algebra $\tilde{\mathfrak{g}}^{(r)}$ [36] (see table 1), the matrices $\tilde{S}$ in (3.8) are nothing but the modular transformation matrix between $\mathfrak{g}^{(r)}$ and $\tilde{\mathfrak{g}}^{(r)}$,[4] for which we can set $\eta_\mu = 1$ and obtain

[4] This fact gives an explanation for the coincidence of the twining character $\chi^\omega$ with the ordinary character of the orbit Lie algebra.

| $\mathfrak{g}^{(r)}$ | $A_{2l}^{(2)}$ | $A_{2l-1}^{(2)}$ | $D_{l+1}^{(2)}$ | $E_6^{(2)}$ | $D_4^{(3)}$ |
|---|---|---|---|---|---|
| $\tilde{\mathfrak{g}}^{(r)}$ | $A_{2l}^{(2)}$ | $D_{l+1}^{(2)}$ | $A_{2l-1}^{(2)}$ | $E_6^{(2)}$ | $D_4^{(3)}$ |

**Table 1:** The modular transformation of the twisted affine Lie algebras. $l$ stands for the rank of the algebras.

$\tilde{S}^{(1)T} = \tilde{S}^{(0)}$ [36]. In the following, we restrict ourselves to the case of $\eta_\mu = 1$ and write $\tilde{S}^{(0)}$ as $\tilde{S}$,

$$\tilde{S}^{(0)} = \tilde{S}^{(1)T} \equiv \tilde{S}\,. \tag{3.10}$$

The matrix $\tilde{S}$ satisfies the following properties

$$\begin{aligned} \tilde{S}\tilde{S}^\dagger &= 1\,, \\ (\tilde{S}\tilde{S}^T)_{\mu\nu} &= C_{\mu\nu} = \delta_{\mu\bar{\nu}}, \\ (\tilde{S}^T\tilde{S})_{\tilde{\mu}\tilde{\nu}} &= \widetilde{C}_{\tilde{\mu}\tilde{\nu}} = \delta_{\tilde{\mu}\overline{\tilde{\nu}}} \qquad (\mu,\nu \in \mathcal{I}^\omega,\, \tilde{\mu},\tilde{\nu} \in \tilde{\mathcal{I}})\,, \end{aligned} \tag{3.11}$$

where $C$ and $\widetilde{C}$ are the charge-conjugation matrix of $\mathcal{I}$ and $\tilde{\mathcal{I}}$, respectively.

We turn to the construction of the twisted Cardy states [8, 12]. Suppose that we obtain the twisted Cardy states $|\tilde{\alpha}\rangle_\omega$ in the charge-conjugation modular invariant,

$$|\tilde{\alpha}\rangle_\omega = \sum_{\mu\in\mathcal{I}^\omega} \psi_{\tilde{\alpha}}{}^{\mu} \frac{1}{\sqrt{S_{0\mu}}} |\mu\rangle\!\rangle_\omega\,. \tag{3.12}$$

For $|\tilde{\alpha}\rangle_\omega$ to be well-defined in the charge-conjugation modular invariant, $|\tilde{\alpha}\rangle_\omega$ has to be consistent with the regular Cardy states $|\mu\rangle$ ($\mu \in \mathcal{I}$). From eqs. (3.6) and (3.8), one can calculate the overlap of the twisted states $|\tilde{\alpha}\rangle_\omega$ with the untwisted state $|0\rangle$ ($0 \in \mathcal{I}$),

$$\langle 0|\tilde{q}^{H_c}|\tilde{\alpha}\rangle_\omega = \sum_{\mu\in\mathcal{I}^\omega} \psi_{\tilde{\alpha}}{}^{\mu}\, \chi_\mu^\omega \left(\frac{-1}{\tau}\right) = \sum_{\mu\in\mathcal{I}^\omega,\,\tilde{\mu}\in\tilde{\mathcal{I}}} \psi_{\tilde{\alpha}}{}^{\mu} \tilde{S}_{\mu\tilde{\mu}}\, \chi_{\tilde{\mu}} \left(\frac{\tau}{r}\right). \tag{3.13}$$

Clearly, this amplitude is made to be well-defined if one sets $\psi = \tilde{S}^{-1}$ or $\tilde{S}^T$. Here we take the latter choice $\psi = \tilde{S}^T$,

$$|\tilde{\mu}\rangle_\omega = \sum_{\mu\in\mathcal{I}^\omega} \tilde{S}_{\tilde{\mu}\mu} \frac{1}{\sqrt{S_{0\mu}}} |\mu\rangle\!\rangle_\omega \qquad (\tilde{\mu} \in \tilde{\mathcal{I}})\,, \tag{3.14}$$

for which the overlap with $|0\rangle$ reads

$$\langle 0|\tilde{q}^{H_c}|\tilde{\mu}\rangle_\omega = \sum_{\tilde{\nu}\in\tilde{\mathcal{I}}} (\tilde{S}^T\tilde{S})_{\tilde{\mu}\tilde{\nu}} \chi_{\tilde{\nu}} \left(\frac{\tau}{r}\right) = \chi_{\overline{\tilde{\mu}}} \left(\frac{\tau}{r}\right). \tag{3.15}$$

The twisted Cardy states are therefore labeled by the representations $\tilde{\mu} \in \tilde{\mathcal{I}}$ of the twisted chiral algebra $\mathcal{A}^\omega$.

The twisted state $|\tilde{\mu}\rangle_\omega$ is characterized by the property that the open string spectrum (3.15) between $|\tilde{\mu}\rangle_\omega$ and $|0\rangle$ consists of $\chi_{\tilde{\mu}}$ only. This is completely parallel with Cardy's original construction of the untwisted Cardy states: $|\mu\rangle$ ($\mu \in \mathcal{I}$) is determined so that $\langle\mu|\tilde{q}^{H_c}|0\rangle = \chi_\mu$ holds [1]. From this construction, Cardy has argued that the overlap of the Cardy states can be expressed in terms of the fusion coefficients, namely,

$$\langle\mu|\tilde{q}^{H_c}|\bar{\nu}\rangle = \sum_{\rho\in\mathcal{I}} \mathcal{N}_{\mu\nu}{}^{\rho}\chi_\rho(\tau). \tag{3.16}$$

Substituting $\psi_\mu{}^\nu = S_{\mu\nu}$ in the left-hand side, one obtains the Verlinde formula (2.5). We can proceed in the same way for the twisted states [12]. For simplicity, we restrict ourselves to the case of $r = 2$.[5] Then the counterpart of (3.16) can be written as

$$_\omega\langle\tilde{\mu}|\tilde{q}^{H_c}|\bar{\nu}\rangle = \sum_{\tilde{\rho}\in\tilde{\mathcal{I}}} \mathcal{N}_{\tilde{\mu}\nu}{}^{\tilde{\rho}}\chi_{\tilde{\rho}}\left(\frac{\tau}{r}\right), \tag{3.17a}$$

$$_\omega\langle\tilde{\mu}|\tilde{q}^{H_c}|\bar{\tilde{\nu}}\rangle_\omega = \sum_{\rho\in\mathcal{I}} \mathcal{N}_{\tilde{\mu}\tilde{\nu}}{}^{\rho}\chi_\rho(\tau), \tag{3.17b}$$

where $\mathcal{N}_{\tilde{\mu}\nu}{}^{\tilde{\rho}}$ can be considered as describing the fusion rule of the twisted representation with the ordinary one. Substituting the expression (3.14) for the twisted states, we obtain a generalized Verlinde formula

$$\mathcal{N}_{\tilde{\mu}\tilde{\nu}}{}^{\rho} = \mathcal{N}_{\tilde{\mu}\bar{\rho}}{}^{\bar{\tilde{\nu}}} = \sum_{\sigma\in\mathcal{I}^\omega} \frac{\tilde{S}_{\tilde{\mu}\sigma}\tilde{S}_{\tilde{\nu}\sigma}\overline{S_{\rho\sigma}}}{S_{0\sigma}}. \tag{3.18}$$

Since the fusion coefficients take values in non-negative integers, the above expression (3.17) for the overlaps implies that the mutual consistency of the twisted states (3.14) with the regular states.

The action of the simple currents on the twisted Cardy states (3.14) can be described in terms of the twisted chiral algebra $\mathcal{A}^\omega$. In order to see this, consider the case of an affine Lie algebra $\mathfrak{g}^{(1)}$ and the diagram automorphism $\omega$, for which the coefficient $\tilde{S}_{\tilde{\mu}\nu}$ in the twisted Cardy states is the modular transformation matrix between the twisted affine Lie algebras $\mathfrak{g}^{(r)}$ and $\tilde{\mathfrak{g}}^{(r)}$ (see table 1). The sets, $\mathcal{V},\mathcal{E}$, of the labels of the Cardy states and the Ishibashi states, respectively, read

$$\mathcal{V} = \tilde{\mathcal{I}} = P_+^k(\mathfrak{g}^{(r)}), \qquad \mathcal{E} = \mathcal{I}^\omega \cong P_+^k(\tilde{\mathfrak{g}}^{(r)}). \tag{3.19}$$

Here we used the equivalence of the twining character of $\mathfrak{g}^{(1)}$ and the ordinary character of the orbit Lie algebra $\tilde{\mathfrak{g}}^{(r)}$ to identify $\mathcal{I}^\omega$ with $P_+^k(\tilde{\mathfrak{g}}^{(r)})$. In this case, the action of the simple currents can be identified with the diagram automorphisms of $\mathfrak{g}^{(r)}$ and $\tilde{\mathfrak{g}}^{(r)}$ [18], namely,

$$G(\mathcal{V}) = \mathrm{Aut}(\mathfrak{g}^{(r)}), \qquad G(\mathcal{E}) \cong \mathrm{Aut}(\tilde{\mathfrak{g}}^{(r)}), \tag{3.20}$$

where we denote by $\mathrm{Aut}(\mathfrak{g}^{(r)})$ the group of the diagram automorphisms of $\mathfrak{g}^{(r)}$. For $\mathfrak{g}^{(r)} = A_{2l}^{(2)}, E_6^{(2)}$ and $D_4^{(3)}$, there is no nontrivial automorphism and $\mathrm{Aut}(\mathfrak{g}^{(r)}) = \{1\}$. For $\mathfrak{g}^{(r)} = A_{2l-1}^{(2)}$ and $D_{l+1}^{(2)}$, there is an automorphism of order 2 and $\mathrm{Aut}(\mathfrak{g}^{(r)}) \cong \mathbb{Z}_2$ (see figure 1).

[5] For the case of $r > 2$, we have to introduce $r-1$ twisted sectors in order to describe the fusion rule of the twisted representations.

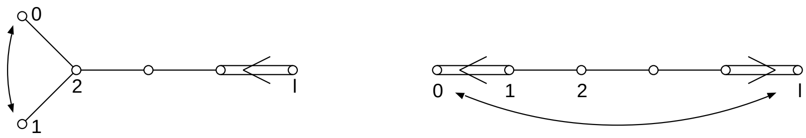


**Figure 1:** The Dynkin diagrams of $A^{(2)}_{2l-1}$ (left) and $D^{(2)}_{l+1}$ (right). The arrows stand for the action of the diagram automorphisms.

Let us apply the foregoing discussion to the case of the $G/H$ theory. Suppose that an automorphism $\omega$ of the horizontal Lie algebra $\mathfrak{g}$ leaves $\mathfrak{h} \subset \mathfrak{g}$ invariant. Then $\omega$ induces an automorphism $\hat{\omega}$ of the $G/H$ theory, which acts on $\hat{\mathcal{I}}$ as

$$\hat{\omega} : (\mu; \nu) \mapsto (\omega\mu; \omega\nu)\,, \qquad (\mu; \nu) \in \hat{\mathcal{I}}\,. \tag{3.21}$$

A natural candidate for the corresponding NIM-rep of $G/H$ can be constructed starting from the twisted Cardy states (3.14) of $G$ and $H$. The formula (2.43) yields the following diagonalization matrix

$$\hat{\psi}_{(\tilde{\mu};\tilde{\nu})}{}^{(\mu;\nu)} = \sqrt{|G_{\rm id}(\mathcal{E})||G_{\rm id}(\mathcal{V})|}\, \tilde{S}^G_{\tilde{\mu}\mu}\tilde{S}^H_{\tilde{\nu}\bar{\nu}}\,, \qquad (\tilde{\mu};\tilde{\nu}) \in \hat{\mathcal{V}}\,,\ (\mu;\nu) \in \hat{\mathcal{E}}\,, \tag{3.22}$$

where $\hat{\mathcal{V}}$ and $\hat{\mathcal{E}}$ are defined as

$$\begin{aligned} \hat{\mathcal{V}} &= \{(\tilde{\mu};\tilde{\nu})\,|\,\tilde{\mu} \in \tilde{\mathcal{I}}^G, \tilde{\nu} \in \tilde{\mathcal{I}}^H,\, \tilde{b}^G_{\tilde{\mu}}(J) = \tilde{b}^H_{\tilde{\nu}}(J')\ (\forall (J,J') \in G_{\rm id}(\mathcal{E}))\,, \\ (J\tilde{\mu}; J'\tilde{\nu}) &= (\tilde{\mu};\tilde{\nu})\ (\forall (J,J') \in G_{\rm id}(\mathcal{V}))\}\,, \end{aligned} \tag{3.23}$$

$$\begin{aligned} \hat{\mathcal{E}} &= \{(\mu;\nu)\,|\,\mu \in \mathcal{I}^{G,\omega}, \nu \in \mathcal{I}^{H,\omega},\, b^G_{\mu}(J) = b^H_{\nu}(J')\ (\forall (J,J') \in G_{\rm id}(\mathcal{V}))\,, \\ (J\mu; J'\nu) &= (\mu;\nu)\ (\forall (J,J') \in G_{\rm id}(\mathcal{E}))\}\,. \end{aligned} \tag{3.24}$$

Since the spectrum $\hat{\mathcal{E}}$ of the Ishibashi states consists of the elements invariant under $\hat{\omega}$, one can expect that this NIM-rep describes the $\hat{\omega}$-twisted Cardy states of $G/H$. As we shall see below, this is the case if $\hat{\psi}$ in (3.22) gives a well-defined NIM-rep.

We begin our discussion with the relation of the spectrum $\hat{\mathcal{E}}$ with the set $\hat{\mathcal{I}}^{\hat{\omega}} \subset \hat{\mathcal{I}}$ of all the representations fixed by $\hat{\omega}$,

$$\hat{\mathcal{I}}^{\hat{\omega}} = \{(\mu;\nu) \in \hat{\mathcal{I}}\,|\,(\omega\mu;\omega\nu) = (\mu;\nu)\}\,. \tag{3.25}$$

Clearly, $\hat{\mathcal{E}}$ is contained in $\hat{\mathcal{I}}^{\hat{\omega}}$; the converse is in general not true. Actually, it is possible that $(\omega\mu;\omega\nu)$ gives rise to the same state as $(\mu;\nu)$ due to the field identification even if $\omega\mu \neq \mu$ (or $\omega\nu \neq \nu$). Therefore $\hat{\mathcal{E}}$ does not necessarily coincide with $\hat{\mathcal{I}}^{\hat{\omega}}$; rather a subset of $\hat{\mathcal{I}}^{\hat{\omega}}$ in general. If $\hat{\mathcal{E}}$ is a true subset of $\hat{\mathcal{I}}^{\hat{\omega}}$, we obtain several extra Ishibashi states. This is a situation analogous to the case of orbifolds, in which we may have some extra Ishibashi states from the twisted sectors. These extra states can be used to resolve the branes sitting at the fixed points of the orbifold to yield fractional branes. For the construction of the boundary states in the $G/H$ theory, the counterpart of the orbifold fixed points is the brane identification fixed point, and one can expect that the existence of the extra Ishibashi states implies the appearance of the brane identification fixed points, which are to be resolved by the extra states. As we will see later, this is checked for the $so(2n)$ diagonal coset.

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In the rest of this section, we restrict ourselves to the case of $\hat{\mathcal{E}} = \hat{\mathcal{I}}^{\hat{\omega}}$. Then a $\hat{\omega}$-invariant representation $(\mu;\nu) \in \hat{\mathcal{I}}^{\hat{\omega}}$ has a representative such that $\mu \in \mathcal{I}^{G,\omega}, \nu \in \mathcal{I}^{H,\omega}$, and the twining character of $(\mu;\nu)$ can be obtained as the branching function of the twining character of $G$,

$$\chi_{\mu}^{G,\omega} = \sum_{\nu \in \mathcal{I}^{H,\omega}} \chi_{(\mu;\nu)}^{\hat{\omega}} \chi_{\nu}^{H,\omega}, \qquad \mu \in \mathcal{I}^{G,\omega}. \tag{3.26a}$$

Similarly, the branching rule for the twisted chiral algebra can be written as

$$\chi_{\tilde{\mu}}^{G} = \sum_{\tilde{\nu} \in \tilde{\mathcal{I}}^{H}} \chi_{(\tilde{\mu};\tilde{\nu})} \chi_{\tilde{\nu}}^{H}, \qquad \tilde{\mu} \in \tilde{\mathcal{I}}^{G}. \tag{3.26b}$$

For an affine Lie algebra $\mathfrak{g}^{(1)}$ and its diagram automorphism $\omega$ of order $r$, the above equations can be regarded as the branching rule of the orbit Lie algebra $\tilde{\mathfrak{g}}^{(r)}$ and the twisted Lie algebra $\mathfrak{g}^{(r)}$, respectively. The modular transformation of the twining character $\chi_{(\mu;\nu)}^{\hat{\omega}}$ yields $\chi_{(\tilde{\mu};\tilde{\nu})}$, and vice versa,

$$\chi_{(\mu;\nu)}^{\hat{\omega}}\left(\frac{-1}{\tau}\right) = \sum_{(\tilde{\mu};\tilde{\nu}) \in \hat{\mathcal{V}}} \tilde{S}^{(0)}_{(\mu;\nu)(\tilde{\mu};\tilde{\nu})} \chi_{(\tilde{\mu};\tilde{\nu})}\left(\frac{\tau}{r}\right), \tag{3.27a}$$

$$\chi_{(\tilde{\mu};\tilde{\nu})}\left(\frac{-1}{\tau}\right) = \sum_{(\mu;\nu) \in \hat{\mathcal{E}}} \tilde{S}^{(1)}_{(\tilde{\mu};\tilde{\nu})(\mu;\nu)} \chi_{(\mu;\nu)}^{\hat{\omega}}\left(\frac{\tau}{r}\right). \tag{3.27b}$$

From the branching rules (3.26), one can express these modular transformation matrices in terms of $\tilde{S}^{G}$ and $\tilde{S}^{H}$ in (3.8),

$$\tilde{S}^{(0)}_{(\mu;\nu)(\tilde{\mu};\tilde{\nu})} = |G_{\mathrm{id}}(\mathcal{V})| \tilde{S}^{G(0)}_{\mu\tilde{\mu}} \overline{\tilde{S}^{H(0)}_{\nu\tilde{\nu}}}, \tag{3.28a}$$

$$\tilde{S}^{(1)}_{(\tilde{\mu};\tilde{\nu})(\mu;\nu)} = |G_{\mathrm{id}}(\mathcal{E})| \tilde{S}^{G(1)}_{\tilde{\mu}\mu} \overline{\tilde{S}^{H(1)}_{\tilde{\nu}\nu}}, \tag{3.28b}$$

where the factors, $|G_{\mathrm{id}}(\mathcal{V})|$, $|G_{\mathrm{id}}(\mathcal{E})|$, arise from rewriting the sum over $\hat{\mathcal{V}}$ and $\hat{\mathcal{E}}$, respectively,

$$\sum_{(\tilde{\mu};\tilde{\nu}) \in \hat{\mathcal{V}}} \to \frac{1}{|G_{\mathrm{id}}(\mathcal{V})|} \sum_{(\tilde{\mu},\tilde{\nu}) \in \tilde{\mathcal{I}}^{G} \otimes \tilde{\mathcal{I}}^{H}} \frac{1}{|G_{\mathrm{id}}(\mathcal{E})|} \sum_{(J,J') \in G_{\mathrm{id}}(\mathcal{E})} \tilde{b}_{\tilde{\mu}}^{G}(J) \tilde{b}_{\tilde{\nu}}^{H}(J')^{-1}, \tag{3.29a}$$

$$\sum_{(\mu;\nu) \in \hat{\mathcal{E}}} \to \frac{1}{|G_{\mathrm{id}}(\mathcal{E})|} \sum_{(\mu,\nu) \in \mathcal{I}^{G,\omega} \otimes \mathcal{I}^{H,\omega}} \frac{1}{|G_{\mathrm{id}}(\mathcal{V})|} \sum_{(J,J') \in G_{\mathrm{id}}(\mathcal{V})} b_{\mu}^{G}(J) b_{\nu}^{H}(J')^{-1}. \tag{3.29b}$$

The expression (3.28) together with the relation (3.9) between $\tilde{S}^{(0)}$ and $\tilde{S}^{(1)}$ implies that the order of the identification groups for $\mathcal{V}$ and $\mathcal{E}$ should take the same value $N$,

$$|G_{\mathrm{id}}(\mathcal{V})| = |G_{\mathrm{id}}(\mathcal{E})| \equiv N. \tag{3.30}$$

Two modular transformation matrices in (3.28) are therefore related with each other via transposition,

$$\tilde{S}^{(0)}_{(\mu;\nu)(\tilde{\mu};\tilde{\nu})} = \tilde{S}^{(1)}_{(\tilde{\mu};\tilde{\nu})(\mu;\nu)} = N \tilde{S}^{G}_{\mu\tilde{\mu}} \overline{\tilde{S}^{H}_{\nu\tilde{\nu}}}, \tag{3.31}$$

where we used the relation (3.10).



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Clearly, the above matrix coincides with the diagonalization matrix (3.22) constructed from the twisted NIM-reps of $G$ and $H$. Hence the states $|(\tilde{\mu};\tilde{\nu})\rangle$ defined in (3.22) are actually the twisted Cardy states (3.14) for the automorphism $\hat{\omega}$ of the $G/H$ theory, and the set $\mathcal{V}$ of the labels for the Cardy states can be identified with those for the branching functions (3.26b) of the twisted chiral algebra.

Applying the argument for the generic twisted states to the present case, we can confirm that the twisted states $|(\tilde{\mu};\tilde{\nu})\rangle$ ($(\tilde{\mu};\tilde{\nu}) \in \hat{\mathcal{V}}$) are consistent with the regular states $|(\mu;\nu)\rangle$ ($(\mu;\nu) \in \hat{\mathcal{I}}$). In particular, the overlaps can be expressed in exactly the same manner as eq. (3.18),

$$\langle(\tilde{\mu};\tilde{\nu})|\tilde{q}^{H_c}|(\bar{\mu};\bar{\nu})\rangle = \sum_{(\tilde{\mu}';\tilde{\nu}')\in\hat{\mathcal{V}}} \hat{\mathcal{N}}_{(\tilde{\mu};\tilde{\nu})(\mu;\nu)}{}^{(\tilde{\mu}';\tilde{\nu}')}\chi_{(\tilde{\mu}';\tilde{\nu}')}\left(\frac{\tau}{r}\right), \tag{3.32a}$$

$$\langle(\tilde{\mu};\tilde{\nu})|\tilde{q}^{H_c}|(\overline{\tilde{\mu}'};\overline{\tilde{\nu}'})\rangle = \sum_{(\mu;\nu)\in\hat{\mathcal{I}}} \hat{\mathcal{N}}_{(\tilde{\mu};\tilde{\nu})(\tilde{\mu}';\tilde{\nu}')}{}^{(\mu;\nu)}\chi_{(\mu;\nu)}(\tau)\,. \tag{3.32b}$$

The fusion coefficients for the twisted representations take the form

$$\begin{aligned}
\hat{\mathcal{N}}_{(\mu;\nu)(\tilde{\mu};\tilde{\nu})}{}^{(\tilde{\mu}';\tilde{\nu}')} &= \sum_{(\rho;\sigma)\in\hat{\mathcal{E}}} \frac{\hat{S}_{(\mu;\nu)(\rho;\sigma)}\tilde{S}_{(\tilde{\mu};\tilde{\nu})(\rho;\sigma)}\overline{\tilde{S}_{(\tilde{\mu}';\tilde{\nu}')(\rho;\sigma)}}}{\hat{S}_{(0;0)(\rho;\sigma)}} \\
&= \sum_{(\rho,\sigma)\in\mathcal{I}^{G,\omega}\otimes\mathcal{I}^{H,\omega}} \sum_{(J,J')\in G_{\mathrm{id}}(\mathcal{V})} b^G_\rho(J)b^H_\sigma(J')^{-1}\frac{S^G_{\mu\rho}\tilde{S}^G_{\tilde{\mu}\rho}\overline{\tilde{S}^G_{\tilde{\mu}'\rho}}}{S^G_{0\rho}}\frac{\overline{S^H_{\nu\sigma}}\,\overline{\tilde{S}^H_{\tilde{\nu}\sigma}}\tilde{S}^H_{\tilde{\nu}'\sigma}}{\overline{S^H_{0\sigma}}} \\
&= \sum_{(J,J')\in G_{\mathrm{id}}(\mathcal{V})} \mathcal{N}^G_{\mu J\tilde{\mu}}{}^{\tilde{\mu}'}\mathcal{N}^H_{\nu J'\tilde{\nu}}{}^{\tilde{\nu}'}\,,
\end{aligned} \tag{3.33}$$

where we used the formula (3.29b) and the relation (3.30). One can see that this expression for the overlaps of the twisted Cardy states coincides with the formula (2.49) for generic NIM-reps of the $G/H$ theory.

## 4. Boundary states in $so(2n)$ diagonal coset theory

The diagonal coset theory $so(2n)_1 \oplus so(2n)_1/so(2n)_2$ has the central charge $c = 1$ and is equivalent to the orbifold $S^1/\mathbb{Z}_2$ at the radius $R = \sqrt{2n}$. In this section, we apply the method given in the previous sections to this particular model to construct the twisted Cardy states following from the diagram automorphisms of $so(2n)$. We will compare the result with the boundary states of $S^1/\mathbb{Z}_2$ in the next section.

### 4.1 Preliminaries

We denote by $\mathcal{I}_k$ the set of all the integrable representations of $so(2n)_k$. For $k = 1, 2$, we obtain

$$\mathcal{I}_1 = \{\Lambda_0, \Lambda_1, \Lambda_{n-1}, \Lambda_n\}, \tag{4.1a}$$

$$\begin{aligned}\mathcal{I}_2 = \{&2\Lambda_0, 2\Lambda_1, 2\Lambda_{n-1}, 2\Lambda_n, \\ &\Lambda_0+\Lambda_{n-1}, \Lambda_0+\Lambda_n, \Lambda_1+\Lambda_{n-1}, \Lambda_1+\Lambda_n, \lambda_j\,(j=1,2,\ldots,n-1)\}\,,\end{aligned} \tag{4.1b}$$

where $\Lambda_i$ is the $i$-th fundamental weight of the affine $so(2n)$ and $\lambda_j$ are defined as

$$\lambda_j = \begin{cases} \Lambda_0 + \Lambda_1 & (j=1)\,, \\ \Lambda_j & (2 \le j \le n-2)\,, \\ \Lambda_{n-1} + \Lambda_n & (j = n-1)\,. \end{cases} \tag{4.2}$$

The simple current group of $so(2n)_k$ consists of four elements

$$G_{\rm sc} = \{J_O = k\Lambda_0, J_V = k\Lambda_1, J_C = k\Lambda_{n-1}, J_S = k\Lambda_n\}\,. \tag{4.3}$$

For even $n$, $G_{\rm sc}$ is factorized as

$$G_{\rm sc} = \{J_O, J_C\} \times \{J_O, J_S\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \quad (n : \text{even})\,, \tag{4.4}$$

while, for odd $n$, $G_{\rm sc}$ is a cyclic group $\mathbb{Z}_4$

$$G_{\rm sc} = \{J_O, J_S, J_V = (J_S)^2, J_C = (J_S)^3\} \cong \mathbb{Z}_4 \quad (n : \text{odd})\,. \tag{4.5}$$

The modular transformation matrices together with the action of the simple currents are found in appendix A.1.

One can construct the primary fields of $so(2n)_1 \oplus so(2n)_1/so(2n)_2$ by applying the selection rule and the field identification to $\mathcal{I}_1 \otimes \mathcal{I}_1 \otimes \mathcal{I}_2$. Since we are considering the diagonal embedding $so(2n)_2 \subset so(2n)_1 \oplus so(2n)_1$, the identification currents are of the form $(J,J,J)\,(J \in G_{\rm sc})$ and $G_{\rm id}$ is isomorphic to $G_{\rm sc}$. There is no fixed point in the field identification and we have $|\hat{\mathcal{I}}| = 4 \times 4 \times (n+7)/(4\times 4) = n+7$ primary fields. We list the resulting spectrum in table 2, in which each field is denoted by a single symbol.[6] The characters of the coset theory can be obtained as the branching function for the embedding $so(2n)_2 \subset so(2n)_1 \oplus so(2n)_1$. We list the explicit form of the characters in appendix B. Since there is no identification fixed point, the modular transformation matrix $\hat{S}$ of the coset theory can be obtained from the formula (2.21),

$$\begin{aligned} \hat{S} =& \frac{1}{\sqrt{8n}} \times \\ & \times \begin{array}{c|ccccc} & u_\pm & \phi_\pm & \chi_s & \sigma_k & \tau_k \\ \hline u_\pm & 1 & 1 & 2 & \pm\sqrt{n} & \pm\sqrt{n} \\ \phi_\pm & 1 & (-1)^n & 2(-1)^s & \pm(-i)^n(-1)^k\sqrt{n} & \pm(-i)^n(-1)^k\sqrt{n} \\ \chi_r & 2 & 2(-1)^r & 4\cos\frac{\pi r s}{n} & 0 & 0 \\ \sigma_j & \pm\sqrt{n} & \pm(-i)^n(-1)^j\sqrt{n} & 0 & (M^n)_{jk}\sqrt{2n} & -(M^n)_{jk}\sqrt{2n} \\ \tau_j & \pm\sqrt{n} & \pm(-i)^n(-1)^j\sqrt{n} & 0 & -(M^n)_{jk}\sqrt{2n} & (M^n)_{jk}\sqrt{2n} \end{array} \\ & \text{where} \quad M = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\pi/4} & e^{i\pi/4} \\ e^{i\pi/4} & e^{-i\pi/4} \end{pmatrix}, \quad M^4 = 1\,. \end{aligned} \tag{4.6}$$

[6]Our notation is the same as that of [37].

| field | coset representation | weight |
|---|---|---|
| $u_+$ | $(\Lambda_0, \Lambda_0; 2\Lambda_0)$ | $0$ |
| $u_-$ | $(\Lambda_0, \Lambda_0; 2\Lambda_1)$ | $1$ |
| $\phi_+$ | $(\Lambda_0, \Lambda_0; 2\Lambda_{n-1})$ $n$ : even; $(\Lambda_0, \Lambda_1; 2\Lambda_{n-1})$ $n$ : odd | $n/4$ |
| $\phi_-$ | $(\Lambda_0, \Lambda_0; 2\Lambda_n)$ $n$ : even; $(\Lambda_0, \Lambda_1; 2\Lambda_n)$ $n$ : odd | $n/4$ |
| $\chi_r$ | $\begin{cases} (\Lambda_0, \Lambda_0; \lambda_r) & r : \text{even} \\ (\Lambda_0, \Lambda_1; \lambda_r) & r : \text{odd} \end{cases}$ | $r^2/4n$ |
| $\sigma_0$ | $(\Lambda_0, \Lambda_{n-1}; \Lambda_0 + \Lambda_{n-1})$ | $1/16$ |
| $\sigma_1$ | $(\Lambda_0, \Lambda_n; \Lambda_0 + \Lambda_n)$ | $1/16$ |
| $\tau_0$ | $(\Lambda_0, \Lambda_{n-1}; \Lambda_1 + \Lambda_n)$ | $9/16$ |
| $\tau_1$ | $(\Lambda_0, \Lambda_n; \Lambda_1 + \Lambda_{n-1})$ | $9/16$ |

**Table 2:** The spectrum of the diagonal coset $so(2n)_1 \oplus so(2n)_1/so(2n)_2$. Here $r = 1, 2, \ldots, n-1$, and there are $n+7$ fields. The last column shows the conformal weight of each field.

### 4.2 Construction of the twisted Cardy states

As we have seen in the previous sections, we can construct the Cardy states, or NIM-reps, of the $G/H$ theory starting from those of the $G$ and $H$ theories. By taking non-trivial NIM-reps of $G$ or $H$, one could in general obtain a non-trivial NIM-rep of $G/H$, besides the regular one. In particular, the Cardy states twisted by an automorphism of the $G/H$ chiral algebra follow from those of the $G$ and $H$ theories. As is argued in section 3, these twisted states are consistent with the regular ones and can be considered as the Cardy states in the charge-conjugation modular invariant.

For the $so(2n)$ diagonal coset theory, the relevant chiral algebras are $\mathfrak{g} = so(2n)_1 \oplus so(2n)_1$ and $\mathfrak{h} = so(2n)_2$. There are three types of automorphisms available for twisting the boundary condition of the coset theory: the order-2 automorphism $\omega_2$ of $so(2n)$, the order-3 automorphism $\omega_3$ of $so(8)$ and the permutation automorphism $\pi$ of $so(2n) \oplus so(2n)$. As we will see below, we can obtain twisted states for $\omega_2$ and $\omega_3$ by applying the construction of the previous sections. For the permutation automorphism $\pi$, however, the resulting NIM-rep suffers from the brane identification fixed points, and we obtain known NIM-reps after the resolution of the fixed points.

#### 4.2.1 $\mathbb{Z}_2$-automorphism of $so(2n)$

The order-2 automorphism $\omega_2$ of $D_n = so(2n)$ exchanges two short legs of the Dynkin diagram of $D_n$,

$$\omega_2 : \Lambda_{n-1} \leftrightarrow \Lambda_n \,, \qquad \Lambda_i \mapsto \Lambda_i \; (i \neq n-1, n) \,. \tag{4.7}$$

From this automorphism, one obtains an automorphism $\omega_2 \oplus \omega_2$ of $\mathfrak{g} = so(2n)_1 \oplus so(2n)_1$, which clearly makes $\mathfrak{h} = so(2n)_2 \subset \mathfrak{g}$ invariant and yields an automorphism $\hat{\omega}_2$ of the coset theory,

$$\hat{\omega}_2 : (\mu_1, \mu_2; \nu) \mapsto (\omega_2\mu_1, \omega_2\mu_2; \omega_2\nu) \,, \qquad (\mu_1, \mu_2; \nu) \in \hat{\mathcal{I}} \,. \tag{4.8}$$

Using the notation in table 2, the action of $\hat{\omega}_2$ can be written as

$$\hat{\omega}_2 : \{u_\pm, \phi_\pm, \chi_r, \sigma_j, \tau_j\} \mapsto \{u_\pm, \phi_\mp, \chi_r, \sigma_{1-j}, \tau_{1-j}\}\,. \tag{4.9}$$

The building blocks of the construction are the twisted NIM-rep of $so(2n)_1$ and $so(2n)_2$. As is explained in section 3 (in particular eq. (3.14)), the diagonalization matrix of the twisted NIM-rep is given by the modular transformation matrix of the twisted chiral algebra, which is $D_n^{(2)}$ in the present case. We list the necessary facts about $D_n^{(2)}$ in appendix A.2.

The NIM-rep for $\hat{\omega}_2$ is obtained from the formula (3.22) by taking $\tilde{S}^G = \tilde{S}_1 \otimes \tilde{S}_1$ and $\tilde{S}^H = \tilde{S}_2$, where $\tilde{S}_k$ stands for the modular transformation matrix of $D_n^{(2)}$ at level $k$. The spectrum $\hat{\mathcal{E}}$ of the Ishibashi states follows from the formula (3.23),

$$\hat{\mathcal{E}} = \{(\Lambda_0, \mu; \nu) \in \hat{\mathcal{I}} \,|\, \mu \in \mathcal{I}_1^{\omega_2}, \nu \in \mathcal{I}_2^{\omega_2}\} = \{u_\pm, \chi_r\; (r = 1, 2, \ldots, n-1)\}\,, \tag{4.10}$$

where we denote by $\mathcal{I}_k^{\omega_2}$ the set of the $\omega_2$-invariant representations of $so(2n)_k$. One can easily confirm from the map (4.9) that $\hat{\mathcal{E}}$ coincides with the set $\hat{\mathcal{I}}^{\hat{\omega}_2}$ of all the $\hat{\omega}_2$-invariant representations and that there is no extra Ishibashi state.

The Cardy states are labeled by the representations of $D_n^{(2)}$. The set $\hat{\mathcal{V}}$ of the Cardy states follows from the formula (3.24),

$$\begin{aligned} \hat{\mathcal{V}} \,&=\, \{(\tilde{\mu}_1, \tilde{\mu}_2; \tilde{\nu}) \,|\, \tilde{\mu}_1, \tilde{\mu}_2 \in \tilde{\mathcal{I}}_1; \tilde{\nu} \in \tilde{\mathcal{I}}_2;\, \tilde{b}_{\tilde{\mu}_1} \tilde{b}_{\tilde{\mu}_2} = \tilde{b}_{\tilde{\nu}}\,; \\ (J\tilde{\mu}_1, J\tilde{\mu}_2; J\tilde{\nu}) \,&=\, (\tilde{\mu}_1, \tilde{\mu}_2; \tilde{\nu})\; (\forall (J, J, J) \in G_{\mathrm{id}}(\mathcal{V}))\}\,, \end{aligned} \tag{4.11}$$

where we denote by $\tilde{I}_k$ the set of the integrable representations of $D_n^{(2)}$ at level $k$. The identification current groups, $G_{\mathrm{id}}(\mathcal{E})$ and $G_{\mathrm{id}}(\mathcal{V})$, can be regarded as the group of the diagram automorphisms of $A_{2n-3}^{(2)}$ and $D_n^{(2)}$, respectively. $\tilde{b}_{\tilde{\mu}}$ in (4.11) is the monodromy charge for the action of $G_{\mathrm{id}}(\mathcal{E})$ (see appendix A.2), whereas $J$ stands for the diagram automorphism of $D_n^{(2)}$. Since there is no fixed point in $\tilde{\mathcal{I}}_1$, we have also no fixed point in the brane identification. Consequently, one can identify $\hat{\mathcal{V}}$ with the set of the labels for branching functions of $D_{n,2}^{(2)} \subset D_{n,1}^{(2)} \oplus D_{n,1}^{(2)}$, which consists of the following elements

$$\hat{\mathcal{V}} = \{(\tilde{\Lambda}_0, \tilde{\Lambda}_0; \tilde{\lambda}_r)\; (r = 0, 1, \ldots, n-1), (\tilde{\Lambda}_0, \tilde{\Lambda}_{n-1}; \tilde{\Lambda}_0 + \tilde{\Lambda}_{n-1})\}\,. \tag{4.12}$$

Here $\tilde{\Lambda}_j$ are the fundamental weights of $D_n^{(2)}$ and $\tilde{\lambda}_j \in \tilde{\mathcal{I}}_2$ are defined in appendix A.2. Putting these things together, the diagonalization matrix (3.22) can be written as follows,

$$\psi^{\hat{\omega}_2} = \frac{1}{\sqrt{2n}} \times \begin{array}{c|cc} & u_\pm & \chi_s \\ \hline (\tilde{\Lambda}_0, \tilde{\Lambda}_0; \tilde{\lambda}_r) & 1 & 2\cos(\frac{\pi}{2n}(2r+1)s) \\ (\tilde{\Lambda}_0, \tilde{\Lambda}_{n-1}; \tilde{\Lambda}_0 + \tilde{\Lambda}_{n-1}) & \pm\sqrt{n} & 0 \end{array}\,. \tag{4.13}$$

We have $n+1$ twisted Cardy states for $\hat{\omega}_2$. The overlaps of these states with the regular ones are expanded into the branching functions of $D_n^{(2)}$ (See appendix B.3 for the explicit form of the branching functions.) For example, the overlap of the twisted state $|(\tilde{\Lambda}_0, \tilde{\Lambda}_0; \tilde{\lambda}_r)\rangle$ with the regular state $|u_+\rangle$ can be written as

$$\langle(\tilde{\Lambda}_0, \tilde{\Lambda}_0; \tilde{\lambda}_r)|\tilde{q}^{H_c}|u_+\rangle = \chi_{(\tilde{\Lambda}_0, \tilde{\Lambda}_0; \tilde{\lambda}_r)}(\tau/2)\,, \tag{4.14}$$

where $\chi_{(\tilde{\Lambda}_0, \tilde{\Lambda}_0; \tilde{\lambda}_r)}$ is the branching function of $D_{n,2}^{(2)} \subset D_{n,1}^{(2)} \oplus D_{n,1}^{(2)}$.



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#### 4.2.2 $\mathbb{Z}_3$-automorphism of $so(8)$

The case of the order-3 automorphism $\omega_3$ of $D_4 = so(8)$ can be treated in the same way as $\omega_2$. The automorphism $\omega_3$ of $D_4$ permutes three legs of the Dynkin diagram of $D_4$,

$$\omega_3 : \Lambda_1 \to \Lambda_3 \to \Lambda_4 \to \Lambda_1 \,. \tag{4.15}$$

This induces an automorphism of the coset theory, which we denote by $\hat{\omega}_3$,

$$\hat{\omega}_3 : (\mu_1, \mu_2; \nu) \mapsto (\omega_3\mu_1, \omega_3\mu_2; \omega_3\nu) \,, \qquad (\mu_1, \mu_2; \nu) \in \hat{\mathcal{I}} \,. \tag{4.16}$$

There are $4 + 7 = 11$ representations for the $so(8)$ coset theory. Using the notation in table 2, the action of $\hat{\omega}_3$ can be written as

$$\hat{\omega}_3 : \begin{Bmatrix} u_+;\ \chi_2; \\ u_-,\ \phi_+,\ \phi_-; \\ \chi_1,\ \sigma_0,\ \sigma_1; \\ \chi_3,\ \tau_0,\ \tau_1 \end{Bmatrix} \mapsto \begin{Bmatrix} u_+;\ \chi_2; \\ \phi_+,\ \phi_-,\ u_-; \\ \sigma_0,\ \sigma_1,\ \chi_1; \\ \tau_0,\ \tau_1,\ \chi_3 \end{Bmatrix} . \tag{4.17}$$

One can see that there are three orbits of length 3 together with two fixed points $u_+, \chi_2$.

The twisted NIM-rep for $\omega_3$ is given by the modular transformation matrix of the twisted chiral algebra $D_4^{(3)}$. We list the necessary facts about $D_4^{(3)}$ in appendix A.3. We can obtain the NIM-rep for $\hat{\omega}_3$ of the coset theory applying the formula (3.22) to $\tilde{S}^G = \tilde{S}_1 \otimes \tilde{S}_1$ and $\tilde{S}^H = \tilde{S}_2$. Here $\tilde{S}_k$ stands for the modular transformation matrix of $D_4^{(3)}$ at level $k$. The spectrum $\hat{\mathcal{E}}$ of the Ishibashi states follows from the formula (3.23),

$$\hat{\mathcal{E}} = \{(\Lambda_0, \mu; \nu) \in \hat{\mathcal{I}} \,|\, \mu \in \mathcal{I}_1^{\omega_3}, \nu \in \mathcal{I}_2^{\omega_3}\} = \{u_+, \chi_2\} \,, \tag{4.18}$$

where we denote by $\mathcal{I}_k^{\omega_3}$ the set of the $\omega_3$-invariant representations of $so(8)_k$. One can see that $\hat{\mathcal{E}}$ coincides with the set $\hat{\mathcal{I}}^{\hat{\omega}_3}$ of all the $\hat{\omega}_3$-invariant representations.

The Cardy states are labeled by the representations of $D_4^{(3)}$. Since there is no non-trivial diagram automorphism for $D_4^{(3)}$, we have no non-trivial identification currents in the present case, i.e. $G_{\rm id}(\mathcal{V}) = G_{\rm id}(\mathcal{E}) = \{1\}$. Then the formula (3.24) yields the set $\mathcal{V}$ in the form

$$\hat{\mathcal{V}} = \{(\tilde{\mu}_1, \tilde{\mu}_2; \tilde{\nu}) \,|\, \tilde{\mu}_1, \tilde{\mu}_2 \in \tilde{\mathcal{I}}_1; \tilde{\nu} \in \tilde{\mathcal{I}}_2\} = \{(\tilde{\Lambda}_0, \tilde{\Lambda}_0; 2\tilde{\Lambda}_0), (\tilde{\Lambda}_0, \tilde{\Lambda}_0; \tilde{\Lambda}_1)\} \,, \tag{4.19}$$

where we denote by $\tilde{I}_k$ the set of the integrable representations of $D_4^{(3)}$ at level $k$ and by $\tilde{\Lambda}_j$ $(j = 0, 1, 2)$ the fundamental weights of $D_4^{(3)}$. $\hat{\mathcal{V}}$ can be regarded as the set of the labels for the branching functions of $D_{4,2}^{(3)} \subset D_{4,1}^{(3)} \oplus D_{4,1}^{(3)}$. The diagonalization matrix (3.22) can be written as follows,

$$\psi^{\hat{\omega}_3} = \frac{1}{\sqrt{2}} \times \begin{array}{c|cc} & u_+ & \chi_2 \\ \hline (\tilde{\Lambda}_0, \tilde{\Lambda}_0; 2\tilde{\Lambda}_0) & 1 & 1 \\ (\tilde{\Lambda}_0, \tilde{\Lambda}_0; \tilde{\Lambda}_1) & 1 & -1 \end{array} . \tag{4.20}$$

We have two twisted Cardy states for $\hat{\omega}_3$. The overlaps of these states with the regular ones are expanded into the branching functions of $D_4^{(3)}$ in the manner parallel to the case of $\hat{\omega}_2$.

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### 4.2.3 Permutation automorphism of $so(2n) \oplus so(2n)$

The permutation automorphism $\pi$ acts on $\mathfrak{g} = so(2n)_1 \oplus so(2n)_1$ as the exchange of two $so(2n)$ factors. Clearly, $\pi$ leaves $\mathfrak{h} = so(2n)_2 \subset \mathfrak{g}$ invariant and induces an automorphism $\hat{\pi}$ of the coset theory,

$$\hat{\pi} : (\mu_1, \mu_2; \nu) \mapsto (\mu_2, \mu_1; \nu)\,, \qquad (\mu_1, \mu_2; \nu) \in \hat{\mathcal{I}}\,. \tag{4.21}$$

In order to obtain the corresponding NIM-rep of the coset theory, we need to know the NIM-rep of $so(2n)_1 \oplus so(2n)_1$ for the boundary condition twisted by the automorphism $\pi$. This problem of finding a NIM-rep for the permutation automorphism of tensor product theories has been solved in [38], which we briefly review.

Suppose that we have a NIM-rep $(n_\mu)_\alpha{}^\beta$ $(\mu \in \mathcal{I}; \alpha, \beta \in \mathcal{V})$ of a fusion algebra $\mathcal{I}$ with fusion coefficients $\mathcal{N}_{\mu\nu}{}^\rho$. Let $\psi_\alpha{}^\rho$ $(\rho \in \mathcal{E})$ be the corresponding diagonalization matrix, $n_\mu = \psi\gamma^{(\mu)}\psi^\dagger$. Then the problem is to find a NIM-rep $n^\pi$ for the tensor product algebra $\mathcal{I} \otimes \mathcal{I}$ with the spectrum $\mathcal{E}^\pi = \{(\mu, \mu) \,|\, \mu \in \mathcal{E}\}$, which consists of the elements fixed by the permutation $\pi : (\mu, \nu) \mapsto (\nu, \mu)$. This is solved by setting

$$n^\pi_{(\mu,\nu)} = n_\mu n_\nu\,, \tag{4.22}$$

where the product in the right-hand side is taken as a $|\mathcal{V}| \times |\mathcal{V}|$ matrix. Since $n_\mu$ form a NIM-rep of $\mathcal{I}$, $n^\pi$ also has entries with non-negative integers. Hence what we have to do is to check that $n^\pi$ satisfy the fusion algebra $\mathcal{I} \otimes \mathcal{I}$, which can be shown as follows,

$$\begin{aligned} n^\pi_{(\mu,\nu)} n^\pi_{(\mu',\nu')} = (n_\mu n_\nu)(n_{\mu'} n_{\nu'}) &= (n_\mu n_{\mu'})(n_\nu n_{\nu'}) \\ &= \sum_{\mu'' \in \mathcal{I}} \mathcal{N}_{\mu\mu'}{}^{\mu''} n_{\mu''} \sum_{\nu'' \in \mathcal{I}} \mathcal{N}_{\nu\nu'}{}^{\nu''} n_{\nu''} \\ &= \sum_{(\mu'',\nu'') \in \mathcal{I}\otimes\mathcal{I}} \mathcal{N}_{\mu\mu'}{}^{\mu''} \mathcal{N}_{\nu\nu'}{}^{\nu''}\, n^\pi_{(\mu'',\nu'')}\,. \end{aligned} \tag{4.23}$$

Here we used the fact that the fusion algebra is commutative to change the order of the product. The diagonalization matrix $\psi^\pi$ for $n^\pi$ is given by $\psi$, which follows immediately from the definition (4.22) of $n^\pi$,

$$\begin{aligned} (n^\pi_{(\mu,\nu)})_\alpha{}^\beta = (n_\mu n_\nu)_\alpha{}^\beta &= (\psi\gamma^{(\mu)}\psi^\dagger\psi\gamma^{(\nu)}\psi^\dagger)_\alpha{}^\beta \\ &= \sum_{\rho \in \mathcal{E}} \psi_\alpha{}^\rho \gamma^{(\mu)}_\rho \gamma^{(\nu)}_\rho \overline{\psi_\beta{}^\rho} \\ &= \sum_{\rho \in \mathcal{E}} \psi_\alpha{}^\rho \gamma^{(\mu,\nu)}_{(\rho,\rho)} \overline{\psi_\beta{}^\rho}\,. \end{aligned} \tag{4.24}$$

This calculation also shows that the spectrum for $n^\pi$ consists of only the elements $(\rho, \rho) \in \mathcal{E} \otimes \mathcal{E}$ fixed by the permutation $\pi$. Hence $n^\pi$ defined in (4.22) is actually a NIM-rep of $\mathcal{I} \otimes \mathcal{I}$ with the spectrum $\mathcal{E}^\pi$.[7] Note that the Cardy states for $n^\pi$ are also labeled by $\mathcal{V}$ for $n$.

[7] In [38], only the case of the regular NIM-rep $n = N$ for $\mathcal{I}$ is considered. Our argument provides its generalization to a generic NIM-rep $n \neq N$.

The action of the simple currents (2.37) on the permutation NIM-rep $n^\pi$ follows from that on $n$. First, the simple current group $G(\mathcal{E}^\pi)$ for the Ishibashi states is the diagonal subgroup of $G(\mathcal{E}) \times G(\mathcal{E})$,

$$J : (\mu, \mu) \mapsto (J\mu, J\mu)\,, \qquad J \in G(\mathcal{E})\,, \tag{4.25}$$

which acts on $\psi^\pi$ as

$$\psi_\alpha^{\pi\,(J\mu,J\mu)} = \psi_\alpha{}^{J\mu} = \tilde{b}_\alpha(J)\psi_\alpha{}^\mu = \tilde{b}_\alpha(J)\psi_\alpha^{\pi\,(\mu,\mu)}\,. \tag{4.26}$$

The action on the Cardy states is given by $(J, J') \in G(\mathcal{V}) \times G(\mathcal{V})$,

$$\psi_\alpha^{\pi\,(\mu,\mu)} b_{(\mu,\mu)}((J,J')) = \psi_\alpha^{\pi\,(\mu,\mu)} b_\mu(J) b_\mu(J') = \psi_\alpha{}^\mu b_\mu(JJ') = \psi_{JJ'\alpha}{}^\mu = \psi^\pi_{JJ'\alpha}{}^{(\mu,\mu)}\,, \tag{4.27}$$

where we used the fact that $J \mapsto b_\mu(J)$ is a representation of the simple current group. This action has a non-trivial stabilizer,

$$\begin{aligned} \mathcal{S}(\mathcal{V}^\pi) &= \{(J,J') \in G(\mathcal{V}) \times G(\mathcal{V}) \,|\, b_\mu(JJ') = 1 \ \forall \mu \in \mathcal{E}\} \\ &= \{(J,J') \in G(\mathcal{V}) \times G(\mathcal{V}) \,|\, JJ' = 1\} \\ &\cong G(\mathcal{V}), \end{aligned} \tag{4.28}$$

and the simple current group $G(\mathcal{V}^\pi)$ for the Cardy states is defined as

$$G(\mathcal{V}^\pi) = G(\mathcal{V}) \times G(\mathcal{V}) / \mathcal{S}(\mathcal{V}^\pi) \cong G(\mathcal{V})\,. \tag{4.29}$$

The above construction of the permutation NIM-rep is easily generalized to the case of tensor products with more than two factors. Namely, a NIM-rep corresponding to the permutation automorphism $\pi$ is obtained as

$$n^\pi_{\mu_1 \otimes \mu_2 \otimes \cdots \otimes \mu_n} = n_{\mu_1} n_{\mu_2} \cdots n_{\mu_n}\,. \tag{4.30}$$

The diagonalization matrix is again given by that for $n$.

Let us apply the above construction to the $so(2n)_1 \oplus so(2n)_1$ part of the coset theory. To make the discussion concrete, we consider only the case of $n = 4$, although the case of $n \neq 4$ can be treated in the same way.

The building blocks of the construction are the permutation NIM-rep (4.22) for $so(8)_1 \oplus so(8)_1$ and the regular NIM-rep for $so(8)_2$, from which one can obtain a NIM-rep (2.43) of the coset theory. Since the spectrum of the permutation NIM-rep consists of the representations of the form $(\mu, \mu)$ $(\mu \in \mathcal{I}_1 = P_+^{k=1}(so(8)))$, the spectrum (2.45) of the Ishibashi states reads

$$\hat{\mathcal{E}} = \{(\Lambda_0, \Lambda_0; \nu) \in \hat{\mathcal{I}}\} = \{u_+, u_-, \phi_+, \phi_-, \chi_2\}\,. \tag{4.31}$$

We therefore obtain five Ishibashi states. The Cardy states of the permutation NIM-rep is also labeled by $\alpha \in \mathcal{I}_1$. Applying the formula (2.44), one obtains

$$\hat{\mathcal{V}} = \{(\alpha;\beta) \,|\, \alpha \in \mathcal{I}_1, \beta \in \mathcal{I}_2,\, b_\alpha(J) = b_\beta(J),\, (J^2\alpha; J\beta) = (\alpha;\beta)\ (\forall J \in G_{\rm sc})\}\,, \tag{4.32}$$

where we used the action of the simple currents (4.27) on the permutation NIM-rep. Using $J^2 = 1$ for $so(8)$, one obtains $\hat{\mathcal{V}}$ explicitly,

$$\hat{\mathcal{V}} = \{(\Lambda_0; 2\Lambda_0)_4, (\Lambda_0; \Lambda_2)_1, (\Lambda_1; \Lambda_0 + \Lambda_1)_2, (\Lambda_3; \Lambda_0 + \Lambda_3)_2, (\Lambda_4; \Lambda_0 + \Lambda_4)_2\}\,. \tag{4.33}$$

The subscript in this equation stands for the length of the brane identification orbit

$$\{(\alpha; J\beta) \,|\, J \in G_{\rm sc}\}\,, \tag{4.34}$$

which is equal to 4 if there is no fixed point. One can see that all the elements other than $(\Lambda_0; 2\Lambda_0)$ are fixed points of the brane identification, and we have to resolve them to obtain a consistent NIM-rep of the coset theory. The degeneracy of the fixed points is given by the ratio of the orbit length to the order of the identification current group. After resolving them, the number of the Cardy states would therefore be $1 + 4 + 2 \times 3 = 11$. Since there are only five Ishibashi states (4.31) in $\hat{\mathcal{E}}$, we have to find $11 - 5 = 6$ states besides those in $\hat{\mathcal{E}}$ for the resolution of the fixed points. One can find these states by considering the action (4.21) of $\hat{\pi}$ on $\hat{\mathcal{I}}$. Actually, one can see that not only the states in $\hat{\mathcal{E}}$ but all the elements in $\hat{\mathcal{I}}$ are invariant under the action of $\hat{\pi}$ due to the field identification. Eventually, we obtain 6 extra Ishibashi states to resolve the fixed points in $\hat{\mathcal{V}}$. The spectrum of the resolved NIM-rep is however the same as that of the regular one. In fact, one can show that the diagonalization matrix (2.43) obtained from the permutation NIM-rep is considered to be the regular NIM-rep after appropriately resolving the fixed points.

One can repeat the same analysis for the case of $n \neq 4$. There are also the brane identification fixed points. After the resolution of them, one again finds nothing new, namely, the regular NIM-rep for even $n$ and the $\hat{\omega}_2$-twisted NIM-rep for odd $n$.

## 5. Relation with orbifold theories

The diagonal coset theory $so(2n)_1 \oplus so(2n)_1/so(2n)_2$ has the central charge $c = c_n^{k=1} + c_n^{k=1} - c_n^{k=2} = 1$, where

$$c_n^k = \frac{(2n^2 - n)k}{k + 2n - 2} \tag{5.1}$$

is the central charge of the SO$(2n)$ WZW model at level $k$. Therefore this model should belong to the class of rational CFTs with $c = 1$, which consists of the circle theory $S^1$ with the radius $R = \sqrt{2n}\,(n \in \mathbb{Z}_{>0})$,[8] its $\mathbb{Z}_2$ orbifold $S^1/\mathbb{Z}_2$ and three exceptional models $\mathbf{T}, \mathbf{O}, \mathbf{I}$ [39]. Actually, by comparing the spectrum, the $so(2n)$ diagonal coset theory is shown to be equivalent to the orbifold theory[9] at $r = \sqrt{n}$

$$\frac{so(2n)_1 \oplus so(2n)_1}{so(2n)_2} \sim S^1(\sqrt{n})/\mathbb{Z}_2\,, \tag{5.2}$$

where we introduced the normalized radius $r \equiv \frac{R}{\sqrt{2}}$ so that the self-dual point corresponds to $r = 1$.

[8] We set $\alpha' = 2$.

[9] To be precise, the charge-conjugation modular invariant of the coset theory yields the orbifold at $r = 1/\sqrt{n}$. Therefore we have to perform the T-duality transformation to obtain the orbifold at $r = \sqrt{n}$. In the following, we always do this to compare the boundary states of the coset theory with those of the orbifold at $r = \sqrt{n}$.

In this section, we use this equivalence of the $so(2n)$ coset theory and the orbifold to check our calculation for the twisted Cardy states in the coset theory.[10] Namely, we shall find that both of the $\mathbb{Z}_2$ and the $\mathbb{Z}_3$ Cardy states have their counterpart in the orbifold theory. The $\mathbb{Z}_2$ states as well as the regular ones are realized by the usual Neumann and Dirichlet states of the orbifold. On the other hand, one can not find the $\mathbb{Z}_3$ states within the Neumann and Dirichlet states; rather the $\mathbb{Z}_3$ states are expressed as the conformal boundary states of the orbifold, which generalize those of $S^1$ constructed in [24, 25]. (The construction of the conformal boundary states for the orbifold (at $r = 2$) is found in appendix D.) This fact assures the validity of our calculation, in particular, the mutual consistency of the twisted states with the regular ones.

### 5.1 Regular Cardy states

We begin with rewriting the regular Cardy states of the coset theory in terms of the orbifold. We review some basic facts about the boundary states in the orbifold in appendix C.

Since the Cardy states are expressed as linear combinations of the Ishibashi states, we need to know the expression of the Ishibashi states in the orbifold theory. From the explicit form of the characters of the $so(2n)$ coset theory (see appendix B.1), we can identify the corresponding Ishibashi states in the orbifold at $r = \sqrt{n}$ as follows,

$$\begin{aligned}
|u_\pm\rangle\rangle &= \frac{1}{2}\sum_{m\in\mathbb{Z}}\left(|\mathcal{D}(2nm)\rangle\rangle \pm |\mathcal{N}(2m)\rangle\rangle\right), \\
|\phi_\pm\rangle\rangle &= \frac{1}{2}\sum_{m\in\mathbb{Z}}\left(|\mathcal{D}(n(2m+1))\rangle\rangle \pm i^n|\mathcal{N}(2m+1)\rangle\rangle\right), \\
|\chi_s\rangle\rangle &= \frac{1}{\sqrt{2}}\sum_{m\in\mathbb{Z}}\left(|\mathcal{D}(2nm+s)\rangle\rangle + |\mathcal{D}(2nm-s)\rangle\rangle\right), \\
|\sigma_j\rangle\rangle &= \frac{1}{2\sqrt{2}}\Big[|\mathcal{D}_T(0)\rangle\rangle + |\mathcal{N}_T(0)\rangle\rangle + i^n(-1)^j\left(|\mathcal{D}_T(\pi)\rangle\rangle + |\mathcal{N}_T(\pi)\rangle\rangle\right)\Big], \\
|\tau_j\rangle\rangle &= \frac{1}{2\sqrt{2}}\Big[|\mathcal{D}_T(0)\rangle\rangle - |\mathcal{N}_T(0)\rangle\rangle + i^n(-1)^j\left(|\mathcal{D}_T(\pi)\rangle\rangle - |\mathcal{N}_T(\pi)\rangle\rangle\right)\Big].
\end{aligned} \tag{5.3}$$

See appendix C for our notation of the Ishibashi states in $S^1/\mathbb{Z}_2$. Then it is straightforward to express the regular Cardy states $|\mu\rangle = \sum_{\nu\in\hat{\mathcal{I}}} \hat{S}_{\mu\nu}\frac{1}{\sqrt{\hat{S}_{0\nu}}}|\nu\rangle\rangle$ in terms of the boundary states of the orbifold [37],

$$\begin{aligned}
|u_\pm\rangle &= |\mathcal{D}[0];\pm\rangle_{\mathcal{O}}, \qquad |\phi_\pm\rangle = |\mathcal{D}[\pi];\pm\rangle_{\mathcal{O}}, \qquad |\chi_s\rangle = |\mathcal{D}[\tfrac{s}{n}\pi]\rangle_{\mathcal{O}}, \\
|\sigma_j\rangle &= |\mathcal{N}[j\pi];+\rangle_{\mathcal{O}}, \qquad |\tau_j\rangle = |\mathcal{N}[j\pi];-\rangle_{\mathcal{O}}.
\end{aligned} \tag{5.4}$$

To be precise, we can multiply each Ishibashi state of the orbifold by some phases in our dictionary (5.3) without changing the normalization. We have fixed this ambiguity by requiring that the regular state $|u_+\rangle$ corresponds to $|\mathcal{D}[0];+\rangle$; one of the fractional D-particles sitting at $x = 0$.

[10] Boundary states in the $c = 1$ models have been studied from the RCFT point of view also in [37].

### 5.2 Twisted Cardy states

#### 5.2.1 $\mathbb{Z}_2$ states

We can identify the $\mathbb{Z}_2$ twisted Cardy states of the $so(2n)$ coset theory with boundary states of the orbifold in the same way as the regular ones. To obtain the expression of the $\mathbb{Z}_2$ twisted Ishibashi states in the orbifold theory, we have two types of conditions. The first one is that the overlap of the twisted states with themselves should yield the ordinary character of the coset theory, which is the same condition as we used in deriving eq. (5.3). The second is that the overlap with the regular states should be equal to the twining character for the $\mathbb{Z}_2$ automorphism, which is the branching function of $A^{(2)}_{2n-3}$. See appendix B.2 for their explicit form. The $\mathbb{Z}_2$ twisted Ishibashi states can be determined from these two conditions together with our expression (5.3) for the regular states. The result is as follows,

$$
\begin{aligned}
|u_\pm\rangle\!\rangle^{\mathbb{Z}_2} &= \frac{1}{2}\sum_{m\in\mathbb{Z}}(-1)^m(|\mathcal{D}(2nm)\rangle\!\rangle \pm |\mathcal{N}(2m)\rangle\!\rangle)\,,\\
|\chi_s\rangle\!\rangle^{\mathbb{Z}_2} &= \frac{1}{\sqrt{2}}\sum_{m\in\mathbb{Z}}(-1)^m(|\mathcal{D}(2nm+s)\rangle\!\rangle + |\mathcal{D}(2nm-s)\rangle\!\rangle)\,.
\end{aligned}
\tag{5.5}
$$

Then one finds that the $\mathbb{Z}_2$ twisted Cardy states (4.13) can be written in the form

$$
|(\tilde\Lambda_0,\tilde\Lambda_0;\tilde\lambda_s)\rangle = |\mathcal{D}\left[\frac{1}{n}\left(s+\frac{1}{2}\right)\pi\right]\rangle_{\mathcal{O}}\,,\qquad |(\tilde\Lambda_0,\tilde\Lambda_{n-1};\tilde\Lambda_0+\tilde\Lambda_{n-1})\rangle = |\mathcal{N}\left[\frac{\pi}{2}\right]\rangle_{\mathcal{O}}\,. \tag{5.6}
$$

Since the right-hand side of these equations coincide with the usual boundary states of the orbifold, the mutual consistency of the $\mathbb{Z}_2$ states with the regular ones is manifest.

#### 5.2.2 $\mathbb{Z}_3$ states

For $n = 4$, we have the $\mathbb{Z}_3$ twisted Cardy states. As we will see shortly, one can not express the $\mathbb{Z}_3$ states within the usual Neumann and Dirichlet states of the orbifold at $r = \sqrt{n} = 2$, and we are led to introduce the conformal boundary states of the orbifold in order to express the $\mathbb{Z}_3$ states.

As we have seen in the last section, there are two $\mathbb{Z}_3$ Cardy states (4.20). For convenience, we introduce a short-hand notation for them,

$$
|0\rangle_{\mathbb{Z}_3} = |(\tilde\Lambda_0,\tilde\Lambda_0;2\tilde\Lambda_0)\rangle\,,\qquad |1\rangle_{\mathbb{Z}_3} = |(\tilde\Lambda_0\,,\tilde\Lambda_0;\tilde\Lambda_1)\rangle\,. \tag{5.7}
$$

Let us consider the overlap of these states with the regular state $|u_\pm\rangle$,

$$
\begin{aligned}
\langle u_\pm|\tilde q^{H_c}|0\rangle_{\mathbb{Z}_3} &= \frac{1}{\eta(q)}\sum_{N\in\mathbb{Z}} q^{(N+1/6)^2}\,,\\
\langle u_\pm|\tilde q^{H_c}|1\rangle_{\mathbb{Z}_3} &= \frac{1}{\eta(q)}\sum_{N\in\mathbb{Z}} q^{(N+1/3)^2}\,,
\end{aligned}
\tag{5.8}
$$

which are readily obtained from the explicit form of these states. From this expression, one can conclude that the $\mathbb{Z}_3$ states are not realized by either the Dirichlet or the Neumann states. In order to see this, note that the spectrum (4.18) of the $\hat{\omega}_3$-invariant Ishibashi states consists of two states $u_+, \chi_2$. The orbifold expression (5.3) of the regular Ishibashi states shows that these two states $u_+, \chi_2$, do not contain the fields in the twisted sectors of the orbifold. Therefore the $\mathbb{Z}_3$ Cardy states (5.7) should consist of only the fields in the untwisted sector, and only the untwisted sector part of $|u_\pm\rangle$, which is given by $\frac{1}{\sqrt{2}}|\mathcal{D}[0]\rangle$, contributes in the overlap (5.8). However, from appendix C, we find that no state of the Dirichlet or the Neumann types has the overlap of the form (5.8) with $\frac{1}{\sqrt{2}}|\mathcal{D}[0]\rangle$. Actually, the overlap of $|u_+\rangle = \frac{1}{\sqrt{2}}|\mathcal{D}[0]\rangle + \cdots$ with non-fractional states takes the form

$$
\begin{aligned}
\langle u_+|\tilde{q}^{H_c}|\mathcal{D}[x]\rangle_{\mathcal{O}} &= \frac{1}{\eta(q)} \sum_{N\in\mathbb{Z}} q^{4(N+\frac{x}{2\pi})^2} \,, \\
\langle u_+|\tilde{q}^{H_c}|\mathcal{N}[\theta]\rangle_{\mathcal{O}} &= \frac{1}{\eta(q)} \sum_{N\in\mathbb{Z}} q^{(N+1/4)^2} \,,
\end{aligned} \tag{5.9}
$$

which is clearly different from that in (5.8). Therefore we can conclude that the $\mathbb{Z}_3$ states (5.7) are not realized within the usual states of the orbifold, which preserve the chiral algebra $u(1)/\mathbb{Z}_2$ of $S^1/\mathbb{Z}_2$. We need states other than those keeping $u(1)/\mathbb{Z}_2$ for describing the $\mathbb{Z}_3$ Cardy states in the orbifold. Actually, in the orbifold, there exist many boundary states besides those of the Neumann and the Dirichlet types. This can be understood from the fact that, in order to obtain a boundary CFT, it may be enough to require that only the Virasoro algebra, rather than its extension, is preserved at the boundaries. In general, finding the conformal boundary states that preserve only the Virasoro algebra is a difficult task since a CFT is irrational with respect to the Virasoro algebra for $c > 1$. For $c = 1$, however, this problem is within the reach of known techniques and has been solved for the case of $S^1$ [24, 25] (see also [26, 27]). This construction of the conformal boundary states is readily generalized to the orbifold $S^1/\mathbb{Z}_2$. We report the detail of the construction in appendix D for the case of $r = 2$.

The construction of the conformal boundary states is based on the fact that the CFT for $S^1$ at $r = 1$ is equivalent to the SU(2) WZW model at level 1. The conformal boundary states of $S^1$ at $r = 1$ are therefore parametrized by SU(2) [24, 25]. Since $S^1/\mathbb{Z}_2$ at $r = n \in \mathbb{Z}_{\geq 1}$ can be considered as an orbifold of the SU(2) WZW model by the discrete subgroup $\Gamma = D_n$ of SU(2),[11] the conformal boundary states of $S^1/\mathbb{Z}_2$ are also parametrized by $g \in \mathrm{SU}(2)$, which we denote by $|g\rangle_\Gamma$ (see appendix D for detail). One can find the $\mathbb{Z}_3$ Cardy states (5.7) within these conformal boundary states of $S^1/\mathbb{Z}_2$. The result is as follows,[12]

$$
\begin{aligned}
|0\rangle_{\mathbb{Z}_3} = |g\rangle_\Gamma \,, \qquad |1\rangle_{\mathbb{Z}_3} &= |-g\rangle_\Gamma \,, \\
g = \frac{1}{2}(1+\gamma_1+\gamma_2+\gamma_3) &= \frac{1}{2}\begin{pmatrix} 1+i & 1+i \\ -1+i & 1-i \end{pmatrix} ,
\end{aligned} \tag{5.10}
$$

[11] To be precise, the dihedral group $D_n$ is a subgroup of $\mathrm{SO}(3) \cong \mathrm{SU}(2)/\{1,-1\}$, where $\{1,-1\}$ is the center of SU(2). As a subgroup of SU(2), we should consider the binary dihedral group $\mathcal{D}_n$.

[12] One can also realize the $\mathbb{Z}_3$ states by setting $g = \frac{1}{2}(1-\gamma_1-\gamma_2-\gamma_3)$. These two choices for $g$ correspond to the twist by $\hat{\omega}_3$ and by $(\hat{\omega}_3)^2$.

where $\gamma_a = i\sigma_a$ form the dihedral group $D_2 = \{1, \gamma_1, \gamma_2, \gamma_3\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. With this choice, one can show that the overlap (5.8) is reproduced by using the formula (D.5) in appendix D. Moreover, the overlap with the $\mathbb{Z}_3$ states themselves

$$
\begin{aligned}
{}_{\mathbb{Z}_3}\langle 0|\tilde{q}^{H_c}|0\rangle_{\mathbb{Z}_3} = {}_{\mathbb{Z}_3}\langle 1|\tilde{q}^{H_c}|1\rangle_{\mathbb{Z}_3} &= \frac{1}{\eta(q)}\left(\sum_{N\in\mathbb{Z}} q^{N^2} + 3\sum_{N\in\mathbb{Z}} q^{(N+1/4)^2}\right), \\
{}_{\mathbb{Z}_3}\langle 0|\tilde{q}^{H_c}|1\rangle_{\mathbb{Z}_3} &= \frac{1}{\eta(q)}\left(\sum_{N\in\mathbb{Z}} q^{(N+1/2)^2} + 3\sum_{N\in\mathbb{Z}} q^{(N+1/4)^2}\right),
\end{aligned} \tag{5.11}
$$

can also be reproduced from our solution (5.10). Hence we can conclude that the $\mathbb{Z}_3$ Cardy states of the $so(8)$ coset theory are realized by the conformal boundary states of the orbifold. This fact again shows the validity of our argument about the mutual consistency of the Cardy states.

## 6. Discussions

In this paper, we have argued the mutual consistency of the twisted boundary states with the untwisted ones in coset CFTs. We have constructed explicitly the twisted Cardy states of the coset theory and have shown that they have well-defined overlaps with the untwisted states. We have also pointed out that the overlap of the twisted states can be described in terms of the branching functions of twisted affine Lie algebras. As a check of our argument, we have given a detailed study of the twisted boundary states in the $so(2n)$ diagonal coset theory and have confirmed their consistency through the equivalence between the coset theory and the orbifold $S^1/\mathbb{Z}_2$. As a by-product, we have found a new type of boundary states in $S^1/\mathbb{Z}_2$, which break $u(1)/\mathbb{Z}_2$ but preserve the Virasoro algebra.

A natural problem related with our result is the issue of the mutual consistency of NIM-reps other than those originating from the automorphisms of the chiral algebra. Actually, there exist many non-trivial NIM-reps for $so(2n)$ at level 2 [11] and we can construct the corresponding NIM-reps of the diagonal coset theory combining them with the NIM-rep of $so(2n)_1$. It is interesting to see whether these NIM-reps are non-trivial or not. If non-trivial, it would be worth studying about their consistency with the regular one.

Our construction of the twisted Cardy states for the triality of $so(8)$ exhibits that the technique of RCFTs can be used for studying the conformal boundary states in the $c = 1$ models. This may be extended to the case of $c > 1$ by considering $su(n)_1$ and its realization by a $(n-1)$-dimensional torus $T^{n-1}$. The role of the Virasoro algebra is taken over by the $W_n$ algebra and we should consider the $W_n$ boundary states instead of the conformal ones. It would be interesting to find an appropriate realization of $T^{n-1}$ as a RCFT equipped with automorphisms and observe whether the corresponding twisted boundary states yield the $W_n$ states.



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## A. Modular transformation matrices of $D_n^{(r)}$

### A.1 $D_n^{(1)}$

$k=1$. There are four integrable representations for $D_{n,1}^{(1)} = so(2n)_1$,

$$\mathcal{I}_1 = P_+^{k=1}(so(2n)) = \{\Lambda_0, \Lambda_1, \Lambda_{n-1}, \Lambda_n\}\,. \tag{A.1}$$

$\Lambda_j$ is the $j$-th fundamental weight of $D_n^{(1)}$. The modular transformation matrix reads

$$\frac{1}{2}\times \begin{array}{c|cccc} & \Lambda_0 & \Lambda_1 & \Lambda_{n-1} & \Lambda_n \\ \hline \Lambda_0 & 1 & 1 & 1 & 1 \\ \Lambda_1 & 1 & 1 & -1 & -1 \\ \Lambda_{n-1} & 1 & -1 & (-i)^n & -(-i)^n \\ \Lambda_n & 1 & -1 & -(-i)^n & (-i)^n \\ \hline\hline \text{class} & O & V & C & S \end{array} \tag{A.2}$$

where the last row shows the conjugacy class of each representation.

$k=2$. There are $n+7$ integrable representations for $so(2n)_2$,

$$\begin{aligned}\mathcal{I}_2 = \{&2\Lambda_0, 2\Lambda_1, 2\Lambda_{n-1}, 2\Lambda_n\,, \\ &\Lambda_0+\Lambda_{n-1}, \Lambda_0+\Lambda_n, \Lambda_1+\Lambda_{n-1}, \Lambda_1+\Lambda_n, \lambda_j\,(j=1,2,\ldots,n-1)\}\,,\end{aligned} \tag{A.3}$$

where $\lambda_j$ are defined as

$$\lambda_j = \begin{cases} \Lambda_0+\Lambda_1 & (j=1)\,, \\ \Lambda_j & (2\le j\le n-2)\,, \\ \Lambda_{n-1}+\Lambda_n & (j=n-1)\,. \end{cases} \tag{A.4}$$

The modular transformation matrix reads

$\frac{1}{\sqrt{8n}} \times$

| | $2\Lambda_0$ | $2\Lambda_1$ | $2\Lambda_{n-1}$ | $2\Lambda_n$ | $\lambda_s$ | $\Lambda_0+\Lambda_{n-1}$ | $\Lambda_0+\Lambda_n$ | $\Lambda_1+\Lambda_n$ | $\Lambda_1+\Lambda_{n-1}$ |
|---|---|---|---|---|---|---|---|---|---|
| $2\Lambda_0$ | 1 | 1 | 1 | 1 | 2 | $\sqrt{n}$ | $\sqrt{n}$ | $\sqrt{n}$ | $\sqrt{n}$ |
| $2\Lambda_1$ | 1 | 1 | 1 | 1 | 2 | $-\sqrt{n}$ | $-\sqrt{n}$ | $-\sqrt{n}$ | $-\sqrt{n}$ |
| $2\Lambda_{n-1}$ | 1 | 1 | $(-1)^n$ | $(-1)^n$ | $2(-1)^s$ | $(-i)^n\sqrt{n}$ | $-(-i)^n\sqrt{n}$ | $(-i)^n\sqrt{n}$ | $-(-i)^n\sqrt{n}$ |
| $2\Lambda_n$ | 1 | 1 | $(-1)^n$ | $(-1)^n$ | $2(-1)^s$ | $-(-i)^n\sqrt{n}$ | $(-i)^n\sqrt{n}$ | $-(-i)^n\sqrt{n}$ | $(-i)^n\sqrt{n}$ |
| $\lambda_r$ | 2 | 2 | $2(-1)^r$ | $2(-1)^r$ | $4\cos\frac{\pi r s}{n}$ | 0 | 0 | 0 | 0 |
| $\Lambda_0+\Lambda_{n-1}$ | $\sqrt{n}$ | $-\sqrt{n}$ | $(-i)^n\sqrt{n}$ | $-(-i)^n\sqrt{n}$ | 0 | $a$ | $b$ | $-a$ | $-b$ |
| $\Lambda_0+\Lambda_n$ | $\sqrt{n}$ | $-\sqrt{n}$ | $-(-i)^n\sqrt{n}$ | $(-i)^n\sqrt{n}$ | 0 | $b$ | $a$ | $-b$ | $-a$ |
| $\Lambda_1+\Lambda_n$ | $\sqrt{n}$ | $-\sqrt{n}$ | $(-i)^n\sqrt{n}$ | $-(-i)^n\sqrt{n}$ | 0 | $-a$ | $-b$ | $a$ | $b$ |
| $\Lambda_1+\Lambda_{n-1}$ | $\sqrt{n}$ | $-\sqrt{n}$ | $-(-i)^n\sqrt{n}$ | $(-i)^n\sqrt{n}$ | 0 | $-b$ | $-a$ | $b$ | $a$ |
| class | $O$ | $O$ | $O(n$:even) $V(n$: odd) | $O(n$:even) $V(n$: odd) | $O(s$:even) $V(s$: odd) | $C$ | $S$ | $C$ | $S$ |

$$\text{where} \quad a = \sqrt{\frac{n}{2}}(1+(-i)^n)\,, \qquad b = \sqrt{\frac{n}{2}}(1-(-i)^n)\,. \tag{A.5}$$

**Simple currents.** For even $n$, the simple current group $G_{\rm sc}$ of $so(2n)_k$ is $\mathbb{Z}_2\times\mathbb{Z}_2$, whose action on the fundamental weights reads

$$\begin{aligned} &J_V : (\Lambda_0,\Lambda_1,\Lambda_2,\ldots,\Lambda_{n-2},\Lambda_{n-1},\Lambda_n) \mapsto (\Lambda_1,\Lambda_0,\Lambda_2,\ldots,\Lambda_{n-2},\Lambda_n,\Lambda_{n-1})\,,\\ &J_S : (\Lambda_0,\Lambda_1,\Lambda_2,\ldots,\Lambda_{n-2},\Lambda_{n-1},\Lambda_n) \mapsto (\Lambda_n,\Lambda_{n-1},\Lambda_{n-2},\ldots,\Lambda_2,\Lambda_1,\Lambda_0)\,,\\ &J_C = J_V J_S\,, \qquad (J_V)^2 = (J_S)^2 = 1\,. \end{aligned} \tag{A.6}$$

For odd $n$, $G_{\rm sc}$ is $\mathbb{Z}_4$ and generated by $J_S$,

$$\begin{aligned} &J_S : (\Lambda_0,\Lambda_1,\Lambda_2,\ldots,\Lambda_{n-2},\Lambda_{n-1},\Lambda_n) \mapsto (\Lambda_n,\Lambda_{n-1},\Lambda_{n-2},\ldots,\Lambda_2,\Lambda_0,\Lambda_1)\,,\\ &J_V = (J_S)^2\,, \qquad J_C = (J_S)^3\,, \qquad (J_S)^4 = 1\,. \end{aligned} \tag{A.7}$$

The monodromy charge $b_\mu(J)$ for $J \in G_{\rm sc}$ is a function of the conjugacy class of $\mu$ and written as

$$b_\mu(J) = \begin{array}{r|cccc} & \mu = O & V & C & S \\ \hline J = J_O & 1 & 1 & 1 & 1 \\ J_V & 1 & 1 & -1 & -1 \\ J_C & 1 & -1 & (-i)^n & -(-i)^n \\ J_S & 1 & -1 & -(-i)^n & (-i)^n \end{array} \tag{A.8}$$

which is nothing but the character table of $G_{\rm sc}$.

### A.2 $D_n^{(2)}$

The modular transformation of the characters of the twisted affine Lie algebra $D_n^{(2)}$ gives rise to those of $A_{2n-3}^{(2)}$, and vice versa. Since the twining characters of $so(2n)_k$ for the $\mathbb{Z}_2$ automorphism $\omega_2$ coincide with the ordinary characters of $A_{2n-3}^{(2)}$ at level $k$, we can label the integrable representations of $A_{2n-3}^{(2)}$ by the $\omega_2$-invariant representations of $so(2n)_k$, which form the set $\mathcal{I}_k^{\omega_2} \subset \mathcal{I}_k$. We therefore use $\mathcal{I}_k^{\omega_2}$ instead of $P_+^k(A_{2n-3}^{(2)})$ and regard the modular transformation matrix $\tilde{S}$ for $(D_n^{(2)}, A_{2n-3}^{(2)})$ as a matrix between $\tilde{\mathcal{I}}_k = P_+^k(D_n^{(2)})$ and $\mathcal{I}_k^{\omega_2}$.

$k = 1$. There are two integrable representations for $k = 1$,

$$\begin{aligned} \tilde{\mathcal{I}}_1 &= \{\tilde{\Lambda}_0, \tilde{\Lambda}_{n-1}\}\,, \\ \mathcal{I}_1^{\omega_2} &= \{\mu \in \mathcal{I}_1 \,|\, \omega_2\mu = \mu\} = \{\Lambda_0, \Lambda_1\}\,, \end{aligned} \tag{A.9}$$

where $\tilde{\Lambda}_j$ $(j = 0, 1, \ldots, n-1)$ are the fundamental weights of $D_n^{(2)}$. The modular transformation matrix reads

$$\frac{1}{\sqrt{2}} \times \quad \begin{array}{c|cc} & \Lambda_0 & \Lambda_1 \\ \hline \tilde{\Lambda}_0 & 1 & 1 \\ \tilde{\Lambda}_{n-1} & 1 & -1 \end{array} \tag{A.10}$$

$k = 2$. There are $n+1$ integrable representations for $k = 2$,

$$\begin{aligned} \tilde{\mathcal{I}}_2 &= \{\tilde{\lambda}_j \; (j = 0, 1, \ldots, n-1), \tilde{\Lambda}_0 + \tilde{\Lambda}_{n-1}\}\,, \\ \mathcal{I}_2^{\omega_2} &= \{\mu \in \mathcal{I}_2 \,|\, \omega_2\mu = \mu\} = \{2\Lambda_0, 2\Lambda_1, \lambda_j \; (j = 1, 2, \ldots, n-1)\}\,, \end{aligned} \tag{A.11}$$

where $\tilde{\lambda}_j$ are defined as

$$\tilde{\lambda}_j = \begin{cases} 2\tilde{\Lambda}_0 & j = 0\,, \\ \tilde{\Lambda}_j & 1 \le j \le n-2\,, \\ 2\tilde{\Lambda}_{n-1} & j = n-1\,. \end{cases} \tag{A.12}$$

The modular transformation matrix reads

$$\frac{1}{\sqrt{2n}} \times \quad \begin{array}{c|ccc} & 2\Lambda_0 & 2\Lambda_1 & \lambda_s \\ \hline \tilde{\lambda}_r & 1 & 1 & 2\cos\left(\frac{\pi}{2n}(2r+1)s\right) \\ \tilde{\Lambda}_0 + \tilde{\Lambda}_{n-1} & \sqrt{n} & -\sqrt{n} & 0 \end{array}\,. \tag{A.13}$$

**Simple currents.** For the modular transformation matrix $\tilde{S}_{\tilde{\mu}\nu}$ $(\tilde{\mu} \in \tilde{\mathcal{I}}_k, \nu \in \mathcal{I}_k^{\omega_2})$, we have two types of actions of simple currents since two sets, $\tilde{\mathcal{I}}_k$ and $\mathcal{I}_k^{\omega_2}$, are distinct. The action on $\tilde{\mathcal{I}}_k$ is caused by the diagram automorphism of $D_n^{(2)}$ (see Fig. 1), which we denote by $\tilde{J}$,

$$\tilde{J} : \tilde{\Lambda}_j \mapsto \tilde{\Lambda}_{n-1-j}\,, \qquad \tilde{J}^2 = 1\,. \tag{A.14}$$

The action on $\mathcal{I}_k^{\omega_2}$ originates from the diagram automorphism of $A_{2n-3}^{(2)}$ and is of order 2. Since $\mathcal{I}_k^{\omega_2} \subset \mathcal{I}_k$, this action can be realized by an element of the simple current group

$G_{\rm sc}$ for $D_n^{(1)}$. Among four elements of $G_{\rm sc}$, only $J_V$ (besides $J_O = 1$) leaves $\mathcal{I}_k^{\omega_2}$ invariant. The action on $\mathcal{I}_k^{\omega_2}$ is therefore caused by $J_V$. Under these actions of simple currents, $\tilde{S}_{\tilde{\mu}\nu}$ transforms as follows [18]

$$\tilde{S}_{J\tilde{\mu}\,\nu} = \tilde{S}_{\tilde{\mu}\nu} b_\nu(J_S)\,, \tag{A.15a}$$

$$\tilde{S}_{\tilde{\mu}\,J_V\nu} = \tilde{b}_{\tilde{\mu}} \tilde{S}_{\tilde{\mu}\nu}\,, \tag{A.15b}$$

where $b_\nu(J_S)$ is the monodromy charge (A.8) for $so(2n)$ and $\tilde{b}_{\tilde{\mu}}$ is defined as

$$\tilde{b}_{\tilde{\mu}} = (-1)^{\tilde{\mu}_{n-1}}\,. \tag{A.16}$$

Here $\tilde{\mu}_{n-1}$ is the Dynkin label of $\tilde{\mu}$. For example, $\tilde{b}_{\tilde{\mu}} = -1$ for only $\tilde{\Lambda}_{n-1}$ in $\tilde{\mathcal{I}}_1$ and $\tilde{\Lambda}_0 + \tilde{\Lambda}_{n-1}$ in $\tilde{\mathcal{I}}_2$.

### A.3 $D_4^{(3)}$

For $so(8)$, besides the $\mathbb{Z}_2$ automorphism $\omega_2$, we have an automorphism $\omega_3$ of order 3 which yields the twisted affine Lie algebra $D_4^{(3)}$. In the same way as the $\mathbb{Z}_2$ case, we can label the integrable representations of $D_4^{(3)}$ also by the $\omega_3$-invariant representations of $so(8)_k$, which form the set $\mathcal{I}_k^{\omega_3} \subset \mathcal{I}_k$. Namely, we have an isomorphism $\mathcal{I}_k^{\omega_3} \cong \tilde{\mathcal{I}}_k^3 = P_+^k(D_4^{(3)})$. Although the modular transformation of the characters of $D_4^{(3)}$ gives rise to those of $D_4^{(3)}$ itself, it is convenient for describing the twisted Cardy states to regard the modular transformation matrix $\tilde{S}$ for $D_4^{(3)}$ as a matrix between $\tilde{\mathcal{I}}_k^3$ and $\mathcal{I}_k^{\omega_3}$.

$k = 1$. There is only one integrable representation for $k = 1$,

$$\begin{aligned} \tilde{\mathcal{I}}_1^3 &= \{\tilde{\Lambda}_0\}\,, \\ \mathcal{I}_1^{\omega_3} &= \{\mu \in \mathcal{I}_1 \,|\, \omega_3\mu = \mu\} = \{\Lambda_0\}\,, \end{aligned} \tag{A.17}$$

where $\tilde{\Lambda}_j$ $(j = 0, 1, 2)$ are the fundamental weights of $D_4^{(3)}$.

$k = 2$. There are two integrable representations for $k = 2$,

$$\begin{aligned} \tilde{\mathcal{I}}_2^3 &= \{2\tilde{\Lambda}_0, \tilde{\Lambda}_1\}\,, \\ \mathcal{I}_2^{\omega_3} &= \{\mu \in \mathcal{I}_2 \,|\, \omega\mu = \mu\} = \{2\Lambda_0, \Lambda_2\}\,. \end{aligned} \tag{A.18}$$

The modular transformation matrix reads

$$\frac{1}{\sqrt{2}} \times \quad \begin{array}{c|cc} & 2\Lambda_0 & \Lambda_2 \\ \hline 2\tilde{\Lambda}_0 & 1 & 1 \\ \tilde{\Lambda}_1 & 1 & -1 \end{array}\,. \tag{A.19}$$

Since there is no diagram automorphism for $D_4^{(3)}$, the simple current group for $\tilde{\mathcal{I}}_k^3$ is trivial.

## B. Characters of $so(2n)_1 \oplus so(2n)_1/so(2n)_2$

### B.1 Ordinary characters

There are $n+7$ independent representations in the branching of $so(2n)_1 \oplus so(2n)_1$ into $so(2n)_2$ as is shown in table 2. The corresponding branching functions are determined by decomposing the product of characters of $so(2n)_1$ into those of $so(2n)_2$ [40] (see §13.15 in [36]):

$$\begin{aligned}
u_\pm(q) &= \frac{1}{\eta(q)}\frac{1}{2}\left(\sum_{N\in\mathbb{Z}} q^{nN^2} \pm \sum_{N\in\mathbb{Z}} (-1)^N q^{N^2}\right), \\
\phi_\pm(q) &= \frac{1}{\eta(q)}\frac{1}{2}\sum_{N\in\mathbb{Z}} q^{n(N+\frac{1}{2})^2}, \\
\chi_r(q) &= \frac{1}{\eta(q)}\sum_{N\in\mathbb{Z}} q^{n(N+\frac{r}{2n})^2} \qquad (r=1,2,\ldots,n-1), \\
\sigma_j(q) &= \frac{1}{\eta(q)}\sum_{N\in\mathbb{Z}} q^{4(N+\frac{1}{8})^2} \qquad (j=0,1), \\
\tau_j(q) &= \frac{1}{\eta(q)}\sum_{N\in\mathbb{Z}} q^{4(N+\frac{3}{8})^2} \qquad (j=0,1).
\end{aligned} \tag{B.1}$$

Here we denote the branching functions by the same symbol as the corresponding representations, e.g. $u_+(q)$ stands for $\chi_{(\Lambda_0,\Lambda_0;2\Lambda_0)}$.

### B.2 Twining characters

#### B.2.1 $\mathbb{Z}_2$-automorphism

The $\mathbb{Z}_2$-automorphism $\omega_2$ of $so(2n)$ yields an automorphism $\hat{\omega}_2$ of the diagonal coset $so(2n)_1 \oplus so(2n)_1/so(2n)_2$, which acts on the representations as

$$\hat{\omega}_2 : (\mu_1, \mu_2; \nu) \mapsto (\omega_2\mu_1, \omega_2\mu_2; \omega_2\nu). \tag{B.2}$$

There are $n+1$ representations invariant under $\hat{\omega}_2$, which form the set $\hat{\mathcal{I}}^{\hat{\omega}_2}$,

$$\hat{\mathcal{I}}^{\hat{\omega}_2} = \{u_\pm, \chi_r\,(r=1,2,\ldots,n-1)\} \subset \hat{\mathcal{I}}. \tag{B.3}$$

Note that all the elements of $\hat{\mathcal{I}}^{\hat{\omega}_2}$ can be written in the form $(\Lambda_0, \mu; \nu), \omega_2\mu = \mu, \omega_2\nu = \nu$. Hence the twining characters for the coset theory can be determined from the following equations

$$\chi^{\omega_2}_{\Lambda_0}\chi^{\omega_2}_{\mu} = \sum_\nu \chi^{\hat{\omega}_2}_{(\Lambda_0,\mu;\nu)}\chi^{\omega_2}_\nu, \tag{B.4}$$

where $\chi^{\omega_2}_\mu$ is the twining character of $so(2n)$. Since $\chi^{\omega_2}_\mu$ coincides with the ordinary character of $A^{(2)}_{2n-3}$, one can regard the above equation as the branching rule of $A^{(2)}_{2n-3,2} \subset A^{(2)}_{2n-3,1} \oplus A^{(2)}_{2n-3,1}$ and $\chi^{\hat{\omega}_2}_{(\Lambda_0,\mu;\nu)}$ as the corresponding branching function, which can be obtained in the same way as $D_n^{(1)}$. The result is written as follows,

$$
\begin{aligned}
u_{\pm}^{\mathbb{Z}_2}(q) \equiv \chi_{u_\pm}^{\hat{\omega}_2}(q) &= \frac{1}{\eta(q)}\frac{1}{2}\left(\sum_{N\in\mathbb{Z}}(-1)^N q^{nN^2} \pm \sum_{N\in\mathbb{Z}}(-1)^N q^{N^2}\right), \\
\chi_r^{\mathbb{Z}_2}(q) \equiv \chi_{\chi_r}^{\hat{\omega}_2}(q) &= \frac{1}{\eta(q)}\sum_{N\in\mathbb{Z}}(-1)^N q^{n(N+\frac{r}{2n})^2} \quad (r=1,2,\dots,n-1)\,.
\end{aligned}
\tag{B.5}
$$

### B.2.2 $\mathbb{Z}_3$-automorphism

The case of the $\mathbb{Z}_3$-automorphism $\omega_3$ for $so(8)$ can be treated in exactly the same way as the $\mathbb{Z}_2$ case discussed above. We denote by $\hat{\omega}_3$ the associated automorphism of the $so(8)$ diagonal coset theory. Among 11 representations in $\hat{\mathcal{I}}$, only two of them are invariant under $\hat{\omega}_3$,

$$
\hat{\mathcal{I}}^{\hat{\omega}_3} = \{u_+, \chi_2\} \subset \hat{\mathcal{I}}\,. \tag{B.6}
$$

The corresponding twining characters of the coset theory can be obtained as the branching function of $D_{4,2}^{(3)} \subset D_{4,1}^{(3)} \oplus D_{4,1}^{(3)}$. The result is written as follows,

$$
\begin{aligned}
u_{+}^{\mathbb{Z}_3}(q) \equiv \chi_{u_+}^{\hat{\omega}_3}(q) &= \frac{1}{\eta(q)}\sum_{N\in\mathbb{Z}} e^{\frac{2\pi i}{3}N} q^{N^2}, \\
\chi_2^{\mathbb{Z}_3}(q) \equiv \chi_{\chi_2}^{\hat{\omega}_3}(q) &= \frac{1}{\eta(q)}\sum_{N\in\mathbb{Z}+\frac{1}{2}} e^{\frac{2\pi i}{3}N} q^{N^2}\,.
\end{aligned}
\tag{B.7}
$$

## B.3 Characters of the twisted chiral algebra

The characters of the twisted chiral algebra for the $\mathbb{Z}_2$-automorphism $\hat{\omega}_2$ are given by the branching functions of $D_{n,2}^{(2)} \subset D_{n,1}^{(2)} \oplus D_{n,1}^{(2)}$, which are determined by

$$
\chi_{\tilde{\Lambda}_0}\chi_{\tilde{\mu}} = \sum_{\tilde{\nu}} \chi_{(\tilde{\Lambda}_0,\tilde{\mu};\tilde{\nu})}\chi_{\tilde{\nu}}, \tag{B.8}
$$

where $\tilde{\mu} \in P_+^{k=1}(D_n^{(2)}), \tilde{\nu} \in P_+^{k=2}(D_n^{(2)})$. There are $n+1$ branching functions and the result is written as follows,

$$
\begin{aligned}
\chi_{(\tilde{\Lambda}_0,\tilde{\Lambda}_0;\tilde{\lambda}_j)}(q) &= \frac{1}{\eta(q^2)}\sum_{N\in\mathbb{Z}} q^{2n(N+\frac{1}{2n}(j+\frac{1}{2}))^2} \qquad (j=0,1,\dots,n-1)\,, \\
\chi_{(\tilde{\Lambda}_0,\tilde{\Lambda}_{n-1};\tilde{\Lambda}_0+\tilde{\Lambda}_{n-1})}(q) &= \frac{1}{\eta(q^2)}\sum_{N\in\mathbb{Z}} q^{2(N+\frac{1}{4})^2}\,,
\end{aligned}
\tag{B.9}
$$

where $\tilde{\lambda}_j$ are the integrable representations for $k=2$ defined in eq. (A.12).

# C. Boundary states in the orbifold $S^1/\mathbb{Z}_2$

## C.1 $S^1$

The circle $S^1$ is expressed by a free boson $X$,[13]

$$
X(z,\bar{z}) = x + ip_L \ln z + ip_R \ln \bar{z} + i\sum_{n\neq 0}\frac{1}{n}a_n z^{-n} + i\sum_{n\neq 0}\frac{1}{n}\tilde{a}_n \bar{z}^{-n}\,. \tag{C.1}
$$

[13] We set $\alpha' = 2$.

Let $R$ be the radius of $S^1$. We also introduce the normalized radius

$$r \equiv \frac{R}{\sqrt{\alpha'}} = \frac{R}{\sqrt{2}}\,, \tag{C.2}$$

in which the self-dual point corresponds to $r = 1$. Then the spectrum of the left and right momenta $p_L, p_R$, is written as

$$\begin{aligned} p_L &= \frac{n}{R} + \frac{mR}{2} = \frac{1}{\sqrt{2}}\left(\frac{n}{r} + mr\right), \\ p_R &= \frac{n}{R} - \frac{mR}{2} = \frac{1}{\sqrt{2}}\left(\frac{n}{r} - mr\right), \end{aligned} \tag{C.3}$$

where $n$ and $m$ stand for the momentum and the winding number of the string, respectively.

The Dirichlet and Neumann boundary conditions are expressed on the modes as[14]

$$a_n + \tilde{a}_{-n} = 0 \quad (\text{Dirichlet})\,, \tag{C.4}$$

$$a_n - \tilde{a}_{-n} = 0 \quad (\text{Neumann})\,. \tag{C.5}$$

The Ishibashi states for these boundary conditions read

$$|\mathcal{D}(n)\rangle\rangle = \exp\left(\sum_{N>0} \frac{1}{N} a_{-N}\widetilde{a}_{-N}\right) |(n,0)\rangle\,, \tag{C.6}$$

$$|\mathcal{N}(m)\rangle\rangle = \exp\left(-\sum_{N>0} \frac{1}{N} a_{-N}\widetilde{a}_{-N}\right) |(0,m)\rangle\,, \tag{C.7}$$

where $|(n,m)\rangle$ is the ground state with momentum $n$ and winding number $m$. The overlaps of these states take the form

$$\langle\langle\mathcal{D}(n)|\tilde{q}^{H_c}|\mathcal{D}(n')\rangle\rangle = \delta_{n,n'}\frac{\tilde{q}^{\frac{n^2}{4r^2}}}{\eta(\tilde{q})}\,, \tag{C.8}$$

$$\langle\langle\mathcal{N}(m)|\tilde{q}^{H_c}|\mathcal{N}(m')\rangle\rangle = \delta_{m,m'}\frac{\tilde{q}^{\frac{m^2r^2}{4}}}{\eta(\tilde{q})}\,, \tag{C.9}$$

$$\langle\langle\mathcal{D}(n)|\tilde{q}^{H_c}|\mathcal{N}(m)\rangle\rangle = \delta_{n,0}\delta_{m,0}\frac{1}{\eta(\tilde{q})}\sum_{N\in Z}(-1)^N\tilde{q}^{N^2}\,, \tag{C.10}$$

where $\eta(q)$ is Dedekind's eta function and $H_c = \frac{1}{2}(L_0 + \tilde{L}_0 - \frac{1}{12})$ is the closed string hamiltonian. Then one can write the Dirichlet and Neumann boundary states in the form

$$|\mathcal{D}[x]\rangle = 2^{-1/4}\,r^{-1/2}\sum_{n\in\mathbb{Z}} e^{-inx}|\mathcal{D}(n)\rangle\rangle\,, \qquad 0 \le x < 2\pi\,, \tag{C.11}$$

$$|\mathcal{N}[\theta]\rangle = 2^{-1/4}\,r^{1/2}\sum_{m\in\mathbb{Z}} e^{-im\theta}|\mathcal{N}(m)\rangle\rangle\,, \qquad 0 \le \theta < 2\pi\,. \tag{C.12}$$

[14]The zero modes are defined as $a_0 = p_L$ and $\tilde{a}_0 = p_R$.



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$|\mathcal{D}[x]\rangle$ corresponds to a D0-brane sitting at $X = xR$ while $|\mathcal{N}[\theta]\rangle$ is a D1-brane with the Wilson line $\theta$. One can calculate the overlaps of these states from those of the Ishibashi states,

$$\langle\mathcal{D}[x]|\tilde{q}^{H_c}|\mathcal{D}[x']\rangle = \frac{1}{\eta(q)}\sum_{N\in\mathbb{Z}} q^{r^2(N+\frac{x-x'}{2\pi})^2} \tag{C.13}$$

$$\langle\mathcal{N}[\theta]|\tilde{q}^{H_c}|\mathcal{N}[\theta']\rangle = \frac{1}{\eta(q)}\sum_{N\in\mathbb{Z}} q^{\frac{1}{r^2}(N+\frac{\theta-\theta'}{2\pi})^2} \tag{C.14}$$

$$\langle\mathcal{D}[x]|\tilde{q}^{H_c}|\mathcal{N}[\theta]\rangle = \frac{1}{\eta(q)}\sum_{N\in\mathbb{Z}} q^{(N+\frac{1}{4})^2}\,. \tag{C.15}$$

### C.2 $S^1/\mathbb{Z}_2$

The boundary states of the orbifold $S^1/\mathbb{Z}_2$ follow from those of $S^1$ by the standard procedure,

$$|\mathcal{D}[x]\rangle_{\mathcal{O}} = \frac{1}{\sqrt{2}}(|\mathcal{D}[x]\rangle + |\mathcal{D}[-x]\rangle)\,, \qquad x \neq 0,\, \pi\,, \tag{C.16}$$

$$|\mathcal{N}[\theta]\rangle_{\mathcal{O}} = \frac{1}{\sqrt{2}}(|\mathcal{N}[\theta]\rangle + |\mathcal{N}[-\theta]\rangle)\,, \qquad \theta \neq 0,\, \pi\,. \tag{C.17}$$

The fixed points of the $\mathbb{Z}_2$ action $X \mapsto -X$ can be resolved taking into account the Ishibashi states from the twisted sectors [41],

$$|\mathcal{D}[0];\pm\rangle_{\mathcal{O}} = \frac{1}{\sqrt{2}}|\mathcal{D}[0]\rangle \pm 2^{-1/4}|\mathcal{D}_T(0)\rangle\!\rangle\,, \tag{C.18}$$

$$|\mathcal{D}[\pi];\pm\rangle_{\mathcal{O}} = \frac{1}{\sqrt{2}}|\mathcal{D}[\pi]\rangle \pm 2^{-1/4}|\mathcal{D}_T(\pi)\rangle\!\rangle\,, \tag{C.19}$$

$$|\mathcal{N}[0];\pm\rangle_{\mathcal{O}} = \frac{1}{\sqrt{2}}|\mathcal{N}[0]\rangle \pm 2^{-1/4}\frac{1}{\sqrt{2}}(|\mathcal{N}_T(0)\rangle\!\rangle + |\mathcal{N}_T(\pi)\rangle\!\rangle)\,, \tag{C.20}$$

$$|\mathcal{N}[\pi];\pm\rangle_{\mathcal{O}} = \frac{1}{\sqrt{2}}|\mathcal{N}[\pi]\rangle \pm 2^{-1/4}\frac{1}{\sqrt{2}}(|\mathcal{N}_T(0)\rangle\!\rangle - |\mathcal{N}_T(\pi)\rangle\!\rangle)\,. \tag{C.21}$$

Here we introduced four Ishibashi states associated with the fixed points $x = 0, \pi$,

$$|\mathcal{D}_T(x)\rangle\!\rangle = \exp\left[\sum_{N=\frac{1}{2},\frac{3}{2},\ldots}\frac{1}{N}a_{-N}\widetilde{a}_{-N}\right]|x\rangle_T\,, \tag{C.22}$$

$$|\mathcal{N}_T(x)\rangle\!\rangle = \exp\left[-\sum_{N=\frac{1}{2},\frac{3}{2},\ldots}\frac{1}{N}a_{-N}\widetilde{a}_{-N}\right]|x\rangle_T\,, \tag{C.23}$$

where $|x\rangle_T$ $(x = 0, \pi)$ are the ground states of the twisted sectors. The overlaps of these states read

$$\langle\!\langle\mathcal{D}_T(x)|\tilde{q}^{H_c}|\mathcal{D}_T(x')\rangle\!\rangle = \langle\!\langle\mathcal{N}_T(x)|\tilde{q}^{H_c}|\mathcal{N}_T(x')\rangle\!\rangle = \delta_{x,x'}\frac{1}{\eta(\tilde{q})}\sum_{N\in\mathbb{Z}}\tilde{q}^{(N+\frac{1}{4})^2}\,, \tag{C.24}$$

$$\langle\!\langle\mathcal{D}_T(x)|\tilde{q}^{H_c}|\mathcal{N}_T(x')\rangle\!\rangle = \delta_{x,x'}\frac{1}{\eta(\tilde{q})}\sum_{N\in\mathbb{Z}}(-1)^N\tilde{q}^{(N+\frac{1}{4})^2}\,. \tag{C.25}$$

## D. Conformal boundary states in $S^1/\mathbb{Z}_2$

### D.1 Conventions

We denote by $|j;m,n\rangle\rangle$ the Ishibashi states for the degenerate representation $(j;m,n)$ of the Virasoro algebra with $c=1$ [25]. $|j;m,n\rangle\rangle$ is normalized as follows,

$$\langle\langle j;m,n|q^{H_c}|j';m',n'\rangle\rangle = \delta_{jj'}\delta_{mm'}\delta_{nn'}\chi_j(q)\,, \qquad \chi_j(q) = \frac{1}{\eta(q)}(q^{j^2} - q^{(j+1)^2}). \tag{D.1}$$

The $u(1)$ Ishibashi states $|\mathcal{D}(n)\rangle\rangle, |\mathcal{N}(m)\rangle\rangle$ for $r=1$ are expressed as

$$|\mathcal{D}(n)\rangle\rangle = \sum_{j\geq\frac{|n|}{2}} |j;\frac{n}{2},\frac{n}{2}\rangle\rangle\,, \tag{D.2}$$

$$|\mathcal{N}(m)\rangle\rangle = \sum_{j\geq\frac{|m|}{2}} (-1)^{j-\frac{m}{2}}|j;\frac{m}{2},-\frac{m}{2}\rangle\rangle\,. \tag{D.3}$$

Since $S^1$ at $r=1$ is equivalent to the SU(2) WZW model at level 1, the conformal boundary states at $r=1$ are parametrized by $g\in\mathrm{SU}(2)$ [24, 25],

$$|g\rangle = 2^{-1/4}\sum_{j,m,n} D^j_{m,n}(g)|j;m,n\rangle\rangle\,. \tag{D.4}$$

The overlap of these states reads

$$\langle g_1|\tilde{q}^{H_c}|g_2\rangle = \frac{1}{\eta(q)}\sum_{n\in\mathbb{Z}} q^{(n+\frac{\alpha}{2\pi})^2}\,, \tag{D.5}$$

where $\alpha$ is determined by the trace in the fundamental representation of $su(2)$,

$$\mathrm{Tr}_{j=\frac{1}{2}}(g_1^\dagger g_2) = 2\cos\alpha\,. \tag{D.6}$$

Among these conformal boundary states, we can identify the Dirichlet and the Neumann states,

$$\begin{aligned}|\mathcal{D}[\alpha]\rangle &= 2^{-1/4}\sum_{n\in\mathbb{Z}} e^{-in\alpha}|\mathcal{D}(n)\rangle\rangle \\ &= 2^{-1/4}\sum_{n\in\mathbb{Z}} e^{-in\alpha}\sum_{j\geq\frac{|n|}{2}}|j;\frac{n}{2},\frac{n}{2}\rangle\rangle = |g = \left(\begin{smallmatrix} e^{-i\alpha} & 0 \\ 0 & e^{i\alpha}\end{smallmatrix}\right)\rangle\,, \end{aligned}\tag{D.7}$$

$$\begin{aligned}|\mathcal{N}[\beta]\rangle &= 2^{-1/4}\sum_{m\in\mathbb{Z}} e^{-im\beta}|\mathcal{N}(m)\rangle\rangle \\ &= 2^{-1/4}\sum_{m\in\mathbb{Z}} e^{-im\beta}\sum_{j\geq\frac{|m|}{2}}(-1)^{j-\frac{m}{2}}|j;\frac{m}{2},-\frac{m}{2}\rangle\rangle = |g = \left(\begin{smallmatrix} 0 & e^{-i\beta} \\ -e^{i\beta} & 0\end{smallmatrix}\right)\rangle\,. \end{aligned}\tag{D.8}$$

### D.2 Conformal boundary states in $S^1(r=2)/\mathbb{Z}_2$

We can construct the conformal boundary states of the orbifold $S^1/\mathbb{Z}_2$ at the rational radii starting from those of $S^1$. Here we report only the case of $r=2$. The case of $r \neq 2$ can be treated in the similar way [42].

The orbifold $S^1(r=2)/\mathbb{Z}_2$ can be considered as the orbifold of the SU(2) WZW model at level 1 by $\Gamma \subset \mathrm{SU}(2)/\{1,-1\}$,

$$\Gamma = \{1, \gamma_1 = i\sigma_1, \gamma_2 = i\sigma_2, \gamma_3 = i\sigma_3\} \cong D_2 = \mathbb{Z}_2 \times \mathbb{Z}_2 \,. \tag{D.9}$$

Here $\gamma_3$ changes the radius of $S^1$ from 1 to 2, while $\gamma_2$ does the reflection $X \mapsto -X$. $\Gamma$ acts on the boundary states as

$$\gamma : |g\rangle \mapsto |\gamma g \gamma^{-1}\rangle, \quad \gamma \in \Gamma \,. \tag{D.10}$$

The generic boundary states in the orbifold theory can therefore be written in the form

$$|g\rangle_\Gamma = \frac{1}{\sqrt{|\Gamma|}} \sum_{\gamma \in \Gamma} |\gamma g \gamma^{-1}\rangle, \quad g \in \mathrm{SU}(2)\,, \tag{D.11}$$

unless $g$ belongs to the fixed points of $\Gamma$.

We have three types of fixed points $F_a (a = 1,2,3)$ corresponding to three subgroups $\{1, \gamma_a\} \cong \mathbb{Z}_2$ of $\Gamma$,

$$F_a = \{g \in \mathrm{SU}(2) | \gamma_a g \gamma_a^{-1} = g\}\,. \tag{D.12}$$

We can parametrize $F_a$ as follows,

$$F_1 = \left\{ f_1(\tilde\theta) \equiv \begin{pmatrix} \cos\tilde\theta & i\sin\tilde\theta \\ i\sin\tilde\theta & \cos\tilde\theta \end{pmatrix} | \, 0 \leq \tilde\theta < 2\pi \right\}, \tag{D.13}$$

$$F_2 = \left\{ f_2(\theta) \equiv \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} | \, 0 \leq \theta < 2\pi \right\}, \tag{D.14}$$

$$F_3 = \left\{ f_3(\alpha) \equiv \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix} | \, 0 \leq \alpha < 2\pi \right\}. \tag{D.15}$$

$F_a$'s intersect at $g = \pm 1$,

$$F_1 \cap F_2 \cap F_3 = \{\pm 1\}\,. \tag{D.16}$$

Clearly, $g = \pm 1 \in \mathrm{SU}(2)$ consist of the fixed point set of $\Gamma$.

In order to resolve these fixed points, we need additional Ishibashi states from the twisted sectors. We have three types of twisted sectors corresponding to three types of fixed points $F_a (a = 1,2,3)$. For the fixed points $F_3$, the necessary states are given by the $u(1)$ Ishibashi states at $r = 1$,

$$|\mathcal{T}_3(n)\rangle\!\rangle \equiv \frac{1}{\sqrt{2}} \left( |\mathcal{D}\left(n + \frac{1}{2}\right)\rangle\!\rangle + |\mathcal{D}\left(-n - \frac{1}{2}\right)\rangle\!\rangle \right) \qquad (n \in \mathbb{Z}_{\geq 0})\,. \tag{D.17}$$

The Dirichlet Ishibashi states in this equation coincide with those for $r = 2$, since their momenta take values in half integers in the unit of $r = 1$ (see eq. (C.3)). Then one can resolve $F_3$ to yield the fractional states

$$\begin{aligned}|f_3(\alpha)\rangle_\Gamma &= \frac{1}{2}(|f_3(\alpha)\rangle + |f_3(-\alpha)\rangle) + 2^{1/4}|\mathcal{T}_3[\alpha]\rangle\rangle\,,\\ |\mathcal{T}_3[\alpha]\rangle\rangle &= \sum_{n\in\mathbb{Z}_{\geq 0}} \cos\left(\left(n+\frac{1}{2}\right)\alpha\right)|\mathcal{T}_3(n)\rangle\rangle \qquad (0\leq\alpha<2\pi)\,.\end{aligned} \tag{D.18}$$

One can easily show that these resolved states are nothing but the usual (non-fractional) $D0$-branes in $S^1/\mathbb{Z}_2$ at $r=2$,

$$|f_3(\alpha)\rangle_\Gamma = |\mathcal{D}\left[\frac{\alpha}{2}\right]\rangle_{\mathcal{O}}\,. \tag{D.19}$$

For $F_1$ and $F_2$, the relevant twisted sectors are those coming from the fixed points $X=0,\pi$ of the reflection $X\mapsto -X$. Within the chiral algebra $u(1)/\mathbb{Z}_2$, we have four Ishibashi states from these twisted sectors, namely, $|\mathcal{D}_T(x)\rangle\rangle$ and $|\mathcal{N}_T(x)\rangle\rangle$ ($x=0,\pi$). These states are decomposed into infinite number of the Virasoro Ishibashi states. Actually, the overlaps of these states with themselves are decomposed into the Virasoro characters as follows,

$$\langle\langle\mathcal{D}_T(x)|q^{H_c}|\mathcal{D}_T(x)\rangle\rangle = \frac{1}{\eta(q)}\sum_{n\in\mathbb{Z}} q^{(n+\frac{1}{4})^2} = \sum_{n\in\mathbb{Z}}\chi_{|n+\frac{1}{4}|}\,, \tag{D.20}$$

$$\langle\langle\mathcal{D}_T(x)|q^{H_c}|\mathcal{N}_T(x)\rangle\rangle = \frac{1}{\eta(q)}\sum_{n\in\mathbb{Z}}(-1)^n q^{(n+\frac{1}{4})^2} = \sum_{n\in\mathbb{Z}}(-1)^n\chi_{|n+\frac{1}{4}|}\,. \tag{D.21}$$

Consequently, the Ishibashi states $|\mathcal{D}_T(x)\rangle\rangle, |\mathcal{N}_T(x)\rangle\rangle$ can be written in the form

$$|\mathcal{D}_T(x)\rangle\rangle = \sum_{n\in\mathbb{Z}} ||n+\frac{1}{4}|;x\rangle\rangle\,, \tag{D.22}$$

$$|\mathcal{N}_T(x)\rangle\rangle = \sum_{n\in\mathbb{Z}}(-1)^n ||n+\frac{1}{4}|;x\rangle\rangle\,, \tag{D.23}$$

where $|j;x\rangle\rangle$ is the Virasoro Ishibashi state with dimension $j^2$ from the twisted sector of $X=x$. Using these states, we can write down the Ishibashi states for the resolution of the fixed points $F_1$ and $F_2$,

$$|\mathcal{T}_1(n)\rangle\rangle \equiv \frac{1}{\sqrt{2}}(||\frac{n}{2}+\frac{1}{4}|;0\rangle\rangle - ||\frac{n}{2}+\frac{1}{4}|;\pi\rangle\rangle)\,, \tag{D.24}$$

$$|\mathcal{T}_2(n)\rangle\rangle \equiv \frac{1}{\sqrt{2}}(||\frac{n}{2}+\frac{1}{4}|;0\rangle\rangle + ||\frac{n}{2}+\frac{1}{4}|;\pi\rangle\rangle) \qquad (n\in\mathbb{Z}_{\geq 0})\,. \tag{D.25}$$

These Ishibashi states $|\mathcal{T}_a(n)\rangle\rangle (a=1,2,3; n\in\mathbb{Z}_{\geq 0})$ from the twisted sectors satisfy

$$\langle\langle\mathcal{T}_a(m)|q^{H_c}|\mathcal{T}_b(n)\rangle\rangle = \delta_{ab}\delta_{mn}\frac{1}{\eta(q)}q^{\frac{1}{4}(n+\frac{1}{2})^2}\,. \tag{D.26}$$

Then the resolved boundary states can be constructed in a way parallel to eq. (D.18), namely,

$$\begin{aligned}|f_a(\beta)\rangle_\Gamma &= \frac{1}{2}(|f_a(\beta)\rangle + |f_a(-\beta)\rangle) + 2^{1/4}|\mathcal{T}_a[\beta]\rangle\rangle\,,\\ |\mathcal{T}_a[\beta]\rangle\rangle &= \sum_{n\in\mathbb{Z}_{\geq 0}} \cos\left(\left(n+\frac{1}{2}\right)\beta\right)|\mathcal{T}_a(n)\rangle\rangle \qquad (0\leq\beta<2\pi)\,.\end{aligned} \tag{D.27}$$

Among these states, we can find the fractional $D1$-branes of $S^1(r=2)/\mathbb{Z}_2$,

$$\begin{aligned} |f_1\left(\frac{\pi}{2}\right)\rangle_\Gamma &= |\mathcal{N}[\pi];+\rangle_\mathcal{O}\,, \qquad |f_1\left(\frac{3\pi}{2}\right)\rangle_\Gamma = |\mathcal{N}[\pi];-\rangle_\mathcal{O}\,, \\ |f_2\left(\frac{\pi}{2}\right)\rangle_\Gamma &= |\mathcal{N}[0];+\rangle_\mathcal{O}\,, \qquad |f_2\left(\frac{3\pi}{2}\right)\rangle_\Gamma = |\mathcal{N}[0];-\rangle_\mathcal{O}\,. \end{aligned} \tag{D.28}$$

At $g=\pm1$, we might expect further resolution since all the elements $\gamma\in\Gamma$ fix $g=\pm1$. For $g=1$, we actually need one more resolution and the result is given by the fractional $D0$-branes of the orbifold

$$\begin{aligned} |\mathcal{D}[0];\pm\rangle_\mathcal{O} &= \frac{1}{2}|g=1\rangle + 2^{1/4}\frac{1}{2}\Big(|\mathcal{T}_3[0]\rangle\!\rangle \pm (|\mathcal{T}_2[0]\rangle\!\rangle + |\mathcal{T}_1[0]\rangle\!\rangle)\Big)\,, \\ |\mathcal{D}[\pi];\pm\rangle_\mathcal{O} &= \frac{1}{2}|g=1\rangle + 2^{1/4}\frac{1}{2}\Big(-|\mathcal{T}_3[0]\rangle\!\rangle \pm (|\mathcal{T}_2[0]\rangle\!\rangle + |\mathcal{T}_1[0]\rangle\!\rangle)\Big)\,. \end{aligned} \tag{D.29}$$

For $g=-1$, however, all the fractional branes in (D.27) give the same result

$$|f_1(\pi)\rangle_\Gamma = |f_3(\pi)\rangle_\Gamma = |f_3(\pi)\rangle_\Gamma = |g=-1\rangle\,. \tag{D.30}$$

Clearly, this state can not be decomposed any more and we have no resolution at $g=-1$.

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